

THE PHASE TRANSITION OF DISCREPANCY IN RANDOM HYPERGRAPHS

CALUM MACRURY, TOMÁŠ MASAŘÍK, LEILANI PAI, AND XAVIER PÉREZ-GIMÉNEZ

ABSTRACT. Motivated by the Beck-Fiala conjecture, we study the discrepancy problem in two related models of random hypergraphs on n vertices and m edges. In the first model, each of the m edges is constructed by placing each vertex into the edge independently with probability d/m , where d is a parameter satisfying $d \rightarrow \infty$ and $dn/m \rightarrow \infty$. In the second model, each vertex independently chooses a subset of d edge labels from $[m]$ uniformly at random. Edge i is then defined to be exactly those vertices whose d -subsets include label i .

In the sparse regime, i.e. when $m = O(n)$, we show that with high probability a random hypergraph from either model has discrepancy at least $\Omega(2^{-n/m} \sqrt{dn/m})$.

In the dense regime, i.e. when $m \gg n$, we show that with high probability a random hypergraph from either model has discrepancy at least $\Omega(\sqrt{(dn/m) \log \gamma})$, where $\gamma = \min\{m/n, dn/m\}$. Furthermore, we obtain nearly matching asymptotic upper bounds on the discrepancy. Specifically, we apply the partial colouring lemma of Lovett and Meka to show that, in the dense regime, with high probability the two random hypergraph models each have discrepancy $O(\sqrt{dn/m \log(m/n)})$. In fact, in a significant parameter range we can tighten our analysis to get an upper bound which matches our lower bound up to a constant factor. This result is algorithmic, and together with the work of Bansal and Meka (SODA, 2019), characterizes how the discrepancy of each random hypergraph transitions from $\Theta(\sqrt{d})$ to $o(\sqrt{d})$ as m increases from $m = \Theta(n)$ to $m \gg n$.

1. INTRODUCTION

A **hypergraph**¹ $H = (V, E)$ consists of a set $V = \{v_1, \dots, v_n\}$ of n vertices together with a multiset $E = \{e_1, \dots, e_m\}$ of m edges, where each edge e_i is a subset of V . We denote the size of an edge e as $|e|$. Note that H is allowed to have duplicate edges (i.e. we may have $e_i = e_{i'}$ for some $i \neq i'$). We can bijectively represent H by an $m \times n$ $\{0, 1\}$ -matrix $\mathbf{A} = \mathbf{A}(H) = (A_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, where $A_{i,j} = 1$ if $v_j \in e_i$ and $A_{i,j} = 0$ if $v_j \notin e_i$. We call \mathbf{A} the **incidence matrix** of H . In particular, each pair of duplicate edges in H corresponds to a pair of identical rows in \mathbf{A} . Moreover, we define the **degree** of a vertex v_j of H to be the number of edges containing that vertex, i.e. the number of 1's in the j th column of \mathbf{A} . A classical problem in discrepancy theory is to find a 2-colouring of the vertices of H so that every edge is as “balanced” as possible. To make this more precise, we define a **colouring** of H to be a function $\psi : V \rightarrow \{-1, 1\}$. This can be extended to a map $\psi : E \rightarrow \mathbb{Z}$, by defining $\psi(e) := \sum_{v \in e} \psi(v)$ for each $e \in E$. We call $|\psi(e)|$ the **discrepancy** of edge e , and note that it measures how unbalanced the colouring ψ is on that edge. Further, the **discrepancy** of colouring ψ , denoted $\text{disc}(\psi)$, is the discrepancy of the least balanced edge of H . That is,

$$\text{disc}(\psi) := \max_{e \in E} |\psi(e)|$$

Finally, we define the **discrepancy** of hypergraph H as

$$\text{disc}(H) := \min_{\psi} \text{disc}(\psi),$$

¹The definition of a hypergraph is equivalent to that of a set system, though we exclusively use the former terminology in this work.

where the minimization is over all colourings of V . This definition and other related notions of combinatorial discrepancy have been studied from various angles and in different contexts. (For a more detailed introduction to the subject, we refer to books [20], [9] and [10].)

1.1. An Overview of Worst-Case Results. One of the central problems in discrepancy theory in the above setting is to bound the discrepancy of a hypergraph H in terms of its maximum degree d . In [7], it was proven by Beck and Fiala that the discrepancy of H is no larger than $2d - 1$. Moreover, they conjectured that the correct upper bound is of the order $O(d^{1/2})$. There has been much work in trying to improve on the original bound of [7]. Most recently, it was proven by Bukh [8] that $\text{disc}(H) \leq 2d - \lg^*(d)$, where \lg^* is the binary iterated logarithm. This of course yields no asymptotic improvement in terms of d , but to this date is the strongest upper bound known which solely depends on d . If the upper bound is allowed dependence on the multiple parameters of the hypergraph, then there are results yielding improvements for hypergraphs in the correct range of parameters. For instance, Banaszczyk [3] showed that if $n := |V|$, then $\text{disc}(H) = O(\sqrt{d \log n})$ —a bound which was later made algorithmic by Bansal et al. [4]. Potukuchi [23] proved that, for d -regular H , $\text{disc}(H) = O(\sqrt{d} + \lambda)$, where $\lambda := \max_{v \perp \mathbf{1}, \|v\|_2=1} \|\mathbf{A}v\|_2$ and \mathbf{A} is the incidence matrix of H . Bansal, Laddha, and Vempala [5] have since proven a more general theorem which includes this bound as a special case. Very recently, Presenti and Vladu proved that $\text{disc}(H) = O(\sqrt{d} + \sqrt{\lambda \log n})$, which improves on [5, 23] when $\lambda = \Omega(\log n)$. The upper bounds of [5, 21, 23] are each algorithmic.

In order to find upper bounds which depend solely on the maximum degree, restricted classes of hypergraphs have instead been studied. For example, if $H = (V, E)$ is assumed to be both d -regular and d -uniform (that is, the incidence matrix \mathbf{A} has exactly d 1's in each column and row), then a folklore application of the Lovász local lemma can be used to show that there exists a colouring which achieves discrepancy $O(\sqrt{d \log d})$ (see [12, 22] for details).

1.2. Random Discrepancy Models. Another approach is to restrict one's attention to hypergraphs which are generated randomly. In this work, we focus on two specific random hypergraph models. Both of these models are defined as distributions over the set of hypergraphs with $n \geq 1$ vertices and $m \geq 1$ edges.

1.2.1. The Edge-Independent Model. In [17], Hoberg and Rothvoss introduced a random hypergraph model, denoted $\mathbb{H}(n, m, p)$, in which a probability parameter $0 \leq p \leq 1$ is given (in addition to n and m). Their model, which we refer to as the **edge-independent model**, is a distribution on hypergraphs which we describe through the following randomized procedure:

- Fix the vertex set $V = \{v_1, \dots, v_n\}$.
- For each $1 \leq i \leq m$, construct edge e_i by placing each $v \in V$ in e_i independently with probability p .

We denote $E = \{e_1, \dots, e_m\}$ and define $\mathbb{H}(n, m, p)$ to be the distribution of the random hypergraph $H = (V, E)$. In other words, the entries of the incidence matrix \mathbf{A} of H are independent Bernoulli random variables of parameter p . We write $H \sim \mathbb{H}(n, m, p)$ to indicate that H is drawn from $\mathbb{H}(n, m, p)$.

If $m = m(n)$ and $p = p(n)$ are functions which depend on n , then we say that $\mathbb{H}(n, m, p)$ satisfies a property $Q = Q(n)$ *w.h.p.*, provided that $\mathbb{P}[H(n) \text{ satisfies } Q(n)] \rightarrow 1$ as $n \rightarrow \infty$, where $H = H(n)$ is drawn from $\mathbb{H}(n, m, p)$. Often, we abuse terminology slightly and say that the random hypergraph H satisfies Q *w.h.p.*

Hoberg and Rothvoss showed that, if $n \geq C_1 m^2 \log m$ and $C_1(\log n)/m \leq p \leq 1/2$ for some sufficiently large constant $C_1 > 0$, then $\text{disc}(H) \leq 1$ *w.h.p.* for $H \sim \mathbb{H}(n, m, p)$. A natural question left open by their work is whether or not H continues to have constant discrepancy when n transitions from $C_1 m^2 \log m$ to $\Theta(m \log m)$. Potukuchi [22] provided a positive answer to this question for the special case when $p = 1/2$ by showing that if $n \geq C_2 m \log m$ for $C_2 = (2 \log 2)^{-1}$, then *w.h.p.* $\text{disc}(H) \leq 1$. Recently, Altschuler and Niles-Weed [2] used Stein's method [11] in conjunction with

the second moment method to substantially generalize this result to hold when $p = p(n)$ depends on n . This includes the challenging case when $p(n) \rightarrow 0$.

1.2.2. *The Edge-Dependent Model.* A related model was introduced by Ezra and Lovett in [12]. As before, we fix $n \geq 1$ and $m \geq 1$, yet we now also consider a parameter $d \geq 1$ which satisfies $d \leq m$. The **edge-dependent model**, denoted $\mathcal{H}(n, m, d)$, is again a distribution on hypergraphs, though we describe it through a different randomized procedure:

- Fix vertex set $V = \{v_1, \dots, v_n\}$.
- For each vertex $v \in V$, independently and *u.a.r.* (uniformly at random) draw $I_v \subseteq [m]$ with $|I_v| = d$.
- For each $1 \leq i \leq m$, construct edge e_i by defining $e_i := \{u \in V : i \in I_u\}$.

By setting $E := \{e_1, \dots, e_m\}$, we define $\mathcal{H}(n, m, d)$ to be the distribution of the random hypergraph $H = (V, E)$. In other words, the incidence matrix \mathbf{A} of $H \sim \mathcal{H}(n, m, d)$ is a random $m \times n$ matrix where each column has d ones and $m - d$ zeros. Note that the columns of \mathbf{A} are independent, but the rows are not. We define what it means for $\mathcal{H}(n, m, d)$ to satisfy a property *w.h.p.* in the same way as in the edge-independent model.

Ezra and Lovett showed that, if $m \geq n \geq d \rightarrow \infty$, then *w.h.p.* $\text{disc}(H) = O(\sqrt{d \log d})$. Bansal and Meka [6] later showed that the factor of $\sqrt{\log d}$ is redundant, thereby matching the bound claimed in the Beck-Fiala conjecture. Specifically, they showed that, for the entire range of n and m , $\text{disc}(H) = O(\sqrt{d})$ *w.h.p.*, provided $d = \Omega((\log \log m)^2)$. We note that the result of Potukuchi [23] implies that the Bansal and Meka upper bound in fact holds for all d . In the edge-independent model, the work of Bansal and Meka can be easily modified to yield a $O(\sqrt{pm})$ upper bound, provided the analogous condition $pm = \Omega((\log \log m)^2)$ holds.

In [14], Franks and Saks considered the more general problem of **vector balancing**. Their main result concerns a collection of random matrix models in which the columns are generated independently. In particular, their results apply to the sparse regime ($m \ll n$) of both the random hypergraph models we have discussed. Specifically, they show that if $n = \Omega(m^3 \log^3 m)$, then *w.h.p.* $\text{disc}(H) \leq 2$, provided H is drawn from $\mathbb{H}(n, m, p)$ or $\mathcal{H}(n, m, d)$ for $p = d/m$. Finally, in a very recent work, Turner et al. [25] considered the problem of vector balancing when the entries of the random matrix \mathbf{A} are each distributed as standard Gaussian random variables which are independent and identically distributed (*i.i.d.*). Amongst other results, they showed that the discrepancy of \mathbf{A} is $\Theta(2^{-n/m} \sqrt{n})$ *w.h.p.*, provided $m \ll n$.

1.3. **An Overview of Our Results.** All results proven in this paper are asymptotic with respect to the parameter n , the number of vertices of the model. Thus, we hereby assume that $m = m(n)$, $p = p(n)$ and $d = d(n)$ are functions which depend on n , with $p = d/m$.

While previous results have successfully matched (or improved upon) the conjectured Beck-Fiala bound of \sqrt{d} in the random hypergraph setting, they either apply to a restricted parameter range [12, 14, 17, 22], or do not provide asymptotically tight results for the full parameter range [6, 12]. In particular, when $n/\log n \ll m \ll n$ or $m \gg n$, the correct order of the discrepancy was previously unknown in either model. We provide the first lower bounds which apply to the full parameter range under the mild assumption that both the average edge size $dn/m = pn$ and degree $d = pm$ tend to infinity with n .

Theorem 1.1. *Suppose that H is generated from $\mathbb{H}(n, m, p)$ with $pn \rightarrow \infty$, $pm \rightarrow \infty$ and p bounded away from 1. If $m = O(n)$, then *w.h.p.**

$$\text{disc}(H) = \Omega\left(\max\{2^{-n/m} \sqrt{pn}, 1\}\right),$$

Moreover, if $m \gg n$, then *w.h.p.*

$$\text{disc}(H) = \Omega\left(\sqrt{pn \log \gamma}\right),$$

where $\gamma := \min\{m/n, pn\}$.

Remark 1. This bound complements the upper bound of 1 in [2] for $m \leq Cn/\log n$ where $C = 2\log 2$, but also implies that if $\varepsilon > 0$ is a constant, then H has non-constant discrepancy for $m \geq (C + \varepsilon)n/\log(np)$. Thus, H exhibits a sharp phase transition at $Cn/\log(np)$ for constant p .

We obtain analogous results regarding the edge-dependent model $\mathcal{H}(n, m, d)$:

Theorem 1.2. *Suppose that H is generated from $\mathcal{H}(n, m, d)$ with $dn/m \rightarrow \infty$, $d \rightarrow \infty$, and d/m bounded away from 1. If $m = O(n)$, then w.h.p.*

$$\text{disc}(H) = \Omega \left(\max\left\{2^{-n/m} \sqrt{\frac{dn}{m}}, 1\right\} \right),$$

Moreover, if $m \gg n$, then w.h.p.

$$\text{disc}(H) = \Omega \left(\sqrt{\frac{dn}{m} \log \gamma} \right),$$

where $\gamma := \min\{m/n, dn/m\}$.

Remark 2. The techniques used in [2] do not seem to apply to the edge dependent model. Thus, if $m = O(n^{1/3}/\log n)$, then [14] implies $\text{disc}(H) \leq 2$, however when $n^{1/3}/\log n \ll m \ll n$, $O(\sqrt{d})$ remains the best known upper bound for $\text{disc}(H)$ [6].

Consider the parameter range in which $n/\log n \ll m \ll n$, which we can interpret as a refinement of the sparse regime of $m = O(n)$. In this parameter range, proving the existence of a colouring whose discrepancy matches our lower bounds remains open for either model. This problem is particularly challenging from an algorithmic perspective, as the partial colouring lemma [19] does not appear to be useful in this range of parameters, and this is the main tool used in the literature.

In contrast, in the dense regime of $m \gg n$, we obtain an (almost) tight upper bound for both models. Moreover, our upper bound is algorithmic, and improves upon the $O(\sqrt{d})$ upper bound of [6]. In particular, at worst our upper bound is $O\left(\sqrt{\frac{dn}{m} \log\left(\frac{m}{n}\right)}\right)$, which is significantly smaller than the upper bound of $O(\sqrt{d})$ of [6], as $m \gg n$. Thus, we show that one can actually *beat* the bound of the Beck-Fiala conjecture in the dense regime.

Theorem 1.3. *Assume that H is drawn from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$ with $p = d/m$, and pick any $\beta = \beta(n) \geq 1$ satisfying²*

$$\beta \frac{dn}{m} \geq \log(m/n) \left(\log\left(\frac{dn}{m}\right) + 2 \right)^5. \quad (1)$$

If $m \gg n$, and $dn/m \rightarrow \infty$, then w.h.p.

$$\text{disc}(H) = O \left(\sqrt{\frac{dn}{m} \log\left(\frac{m}{n}\right) \beta} \right).$$

Moreover, whenever this holds, we can find a colouring ψ of H with such a discrepancy in expected polynomial time.

Remark 3. Observe that the smaller we are able to take β , the better upper bound we get. In particular, if $\beta := \log(m/n)$, then (1) and $\beta(n) \geq 1$ are satisfied (for large enough n). Therefore, we obtain at worst the upper bound of $O\left(\sqrt{\frac{dn}{m} \log\left(\frac{m}{n}\right)}\right)$.

²Let us remark at this place that by \log we always mean the natural logarithm. In the proof of this theorem it is convenient to use \lg to denote logarithm of base 2.

Theorem 1.3 provides asymptotically matching bounds for the lower bounds of Theorems 1.1 and 1.2 in a broad range of the dense regime $m \gg n$. For instance, this happens when $d = pm \geq (m/n)^{1+\varepsilon}$, where $\varepsilon > 0$ is a constant, since in that case $\beta := 1$ clearly satisfies (1), and $\log(dn/m) = \Omega(\log(m/n))$, so $\gamma = \Theta(\log(m/n))$.

Corollary 1.4. *Assume that H is drawn from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$, where $p = d/m$. If $m \gg n$ and $d = pm \geq (m/n)^{1+\varepsilon}$ for constant $\varepsilon > 0$, then w.h.p.*

$$\text{disc}(H) = \Theta \left(\sqrt{\frac{dn}{m} \log \left(\frac{m}{n} \right)} \right).$$

Remark 4. This shows that in the dense regime, the main parameter of interest describing the discrepancy of each random hypergraph model is the average edge size, dn/m , opposed to d , the average/maximum degree (depending on the model).

2. PRELIMINARIES

In this section, we first provide some central-limit-type results for sums of independent random variables, which will be used in Section 3 to obtain lower bounds on the discrepancy. In the sequel, we write $N(0, 1)$ to denote a generic random variable distributed as a standard Gaussian with cumulative distribution function

$$\Phi(x) := \mathbb{P}[N(0, 1) \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-t^2/2) dt \quad \text{for each } x \in \mathbb{R}.$$

The Berry-Esseen Theorem (see section XVI.5 in [13]), which we state below for convenience, yields a quantitative form of the Central Limit Theorem for sums of independent random variables with finite third moment.

Theorem 2.1 (Berry-Esseen). *The following holds for some universal constant $c_{uni} > 0$. Let Y_1, \dots, Y_n be independent random variables with $\mathbb{E}[Y_i] = 0$, $\mathbb{E}[Y_i^2] = \sigma_i^2$ and $\mathbb{E}[|Y_i|^3] = \rho_i < \infty$ for all $i \in [n]$. Consider the sum $Y = \sum_{i=1}^n Y_i$, with standard deviation $\sigma = \sqrt{\sum_{i=1}^n \sigma_i^2}$, and let $\rho = \sum_{i=1}^n \rho_i$. Assume $\sigma > 0$. Then,*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[Y/\sigma \leq x] - \Phi(x)| \leq \frac{(c_{uni}/2)\rho}{\sigma^3}.$$

Remark 5. There has been a series of works improving upon the constant c_{uni} , the latest of which by Shevtsova [24] shows that $c_{uni}/2 \leq 0.560$ in our setting. However, since we are only concerned with the asymptotic growth of discrepancy, the precise value of c_{uni} is not important.

Theorem 2.1 and the triangle inequality immediately yield the following corollary:

Corollary 2.2. *For any interval $I \subseteq (-\infty, \infty)$,*

$$|\mathbb{P}[Y/\sigma \in I] - \mathbb{P}[N(0, 1) \in I]| \leq \frac{c_{uni}\rho}{\sigma^3}.$$

We will apply this result to linear combinations of independent Bernoulli's with coefficients in $\{-1, 1\}$. More precisely, let X_1, \dots, X_n be independent random variables with $X_i \sim \text{Ber}(p_i)$ for some $\mathbf{p} = (p_1, \dots, p_n) \in [0, 1]^n$ (where $\text{Ber}(p_i)$ is a Bernoulli of parameter p_i). Given a vector $\mathbf{a} = (a_1, \dots, a_n) \in \{-1, 1\}^n$, consider the sum $S_{\mathbf{a}, \mathbf{p}} := \sum_{k=1}^n a_k X_k$, whose standard deviation we denote by σ . Under these assumptions we obtain the following bound.

Lemma 2.3. *For any bounded interval $[L, R] \subseteq (-\infty, \infty)$,*

$$\mathbb{P}[S_{\mathbf{a}, \mathbf{p}} \in [L, R]] \leq \frac{c_{uni}}{\sigma} + \left(1 - \exp \left(-\frac{(R-L)^2}{2\pi\sigma^2} \right) \right)^{1/2}.$$

(When $\sigma = 0$, the right-hand side of the bound above is simply interpreted as $+\infty$.)

Proof. Let $\mu = \mathbb{E}[S_{\mathbf{a}, \mathbf{p}}] = \sum_{i=1}^n a_i p_i$. Then we centre $S_{\mathbf{a}, \mathbf{p}}$ by defining the random variables $Y_i = a_i(X_i - \mathbb{E}[X_i])$ and setting

$$Y := \sum_{k=1}^n Y_k = S_{\mathbf{a}, \mathbf{p}} - \mu.$$

Observe that Y has the same standard deviation σ as $S_{\mathbf{a}, \mathbf{p}}$, which we assume is non-zero (otherwise the lemma holds trivially). Moreover, $S_{\mathbf{a}, \mathbf{p}} \in [L, R]$ if and only if $Y/\sigma \in [\tilde{L}, \tilde{R}]$, where $\tilde{L} := (L - \mu)/\sigma$ and $\tilde{R} := (R - \mu)/\sigma$. Further,

$$\mathbb{E}[|Y_i|^3] = (1 - p_i)p_i^3 + p_i(1 - p_i)^3 = p_i(1 - p_i)(p_i^2 + (1 - p_i)^2) \leq p_i(1 - p_i),$$

and hence

$$\rho = \sum_{i=1}^n \mathbb{E}[|Y_i|^3] \leq \sum_{i=1}^n p_i(1 - p_i) = \sigma^2.$$

Then, Corollary 2.2 immediately yields

$$\mathbb{P}[S_{\mathbf{a}, \mathbf{p}} \in [L, R]] = \mathbb{P}[Y/\sigma \in [\tilde{L}, \tilde{R}]] \leq \frac{c_{uni}}{\sigma} + \mathbb{P}[N(0, 1) \in [\tilde{L}, \tilde{R}]].$$

To finalize the proof, it only remains to bound $\mathbb{P}[N(0, 1) \in [\tilde{L}, \tilde{R}]]$. In order to do so, we will use the inequality

$$\int_{-t}^t \exp(-x^2/2) dx \leq \sqrt{2\pi(1 - \exp(-2t^2/\pi))} \quad \text{for all } t \in \mathbb{R}, \quad (2)$$

which can be found in [26]. Furthermore, note that $\exp(-x^2/2)$ is an even function, decreasing with x^2 . Combining this fact with (2), we get

$$\begin{aligned} \mathbb{P}[N(0, 1) \in [\tilde{L}, \tilde{R}]] &= \frac{1}{\sqrt{2\pi}} \int_{\tilde{L}}^{\tilde{R}} \exp(-x^2/2) dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-(\tilde{R}-\tilde{L})/2}^{(\tilde{R}-\tilde{L})/2} \exp(-x^2/2) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-(R-L)/2\sigma}^{(R-L)/2\sigma} \exp(-x^2/2) dx \\ &\leq \sqrt{1 - \exp\left(-\frac{(R-L)^2}{2\pi\sigma^2}\right)}, \end{aligned}$$

which concludes the proof of the lemma. \square

Now suppose there exist $0 < p < 1$, $0 < \zeta < 1$ and $0 \leq \varepsilon < 1$ such that

$$\sum_{i=1}^n p_i \geq (1 - \varepsilon)pn \quad \text{and} \quad p_i \leq \zeta \quad \text{for each } i \in [n]. \quad (3)$$

Then we can restate the upper bound of Lemma 2.3 in the following convenient way, which we use as our key tool in proving Theorems 1.1 and 1.2.

Lemma 2.4. *Suppose p_1, \dots, p_n satisfy (3) for some $0 \leq \varepsilon < 1$, $0 < p < 1$ and $0 < \zeta < 1$. Then, for any bounded interval $[L, R] \subseteq (-\infty, \infty)$,*

$$\mathbb{P}[S_{\mathbf{a}, \mathbf{p}} \in [L, R]] \leq \frac{c_{uni}}{\sqrt{(1-\zeta)(1-\varepsilon)np}} + \left(1 - \exp\left(-\frac{(R-L)^2}{2\pi(1-\zeta)(1-\varepsilon)np}\right)\right)^{1/2} \quad (4)$$

$$\leq \frac{c_{uni} + |R-L|/\sqrt{2\pi}}{\sqrt{(1-\zeta)(1-\varepsilon)np}}. \quad (5)$$

Proof. First note that the variance σ^2 of $S_{\mathbf{a}, \mathbf{p}}$ satisfies

$$\sigma^2 = \sum_{i=1}^n p_i(1-p_i) \geq (1-\zeta) \sum_{i=1}^n p_i \geq (1-\zeta)(1-\varepsilon)pn.$$

This bound, used with Lemma 2.3, immediately gives (4). Then (5) follows by applying the inequality $1 - \exp(-x) \leq x$, which holds for every $x \in \mathbb{R}$. \square

In Theorem 1.3, we prove an upper bound on discrepancy in the dense regime ($m \gg n$). In this parameter range, we make use of the **algorithmic partial colouring lemma**, a seminal result of Lovett and Meka [19] later made deterministic by Levy, Ramadas, and Rothvoss [18]. We defer the statement of this result to Lemma 4.1 of Section 4, as it will not be needed until then.

3. LOWER BOUNDING DISCREPANCY

3.1. The Edge-Independent Model. We now return to the setting of hypergraph discrepancy in the context of the edge-independent model $\mathbb{H}(n, m, p)$. Throughout this section, $m = m(n)$, $p = p(n)$ and asymptotic statements are with respect to $n \rightarrow \infty$. We first observe that *w.h.p.* there are some edges containing an odd number of vertices and thus the discrepancy cannot be zero.

Proposition 3.1. *Suppose $H \sim \mathbb{H}(n, m, p)$ with $m \rightarrow \infty$, $pn \rightarrow \infty$ and p bounded away from 1 as $n \rightarrow \infty$. Then *w.h.p.* $\text{disc}(H) \geq 1$.*

Proof. By hypothesis, $p \leq 1 - \nu$ for some sufficiently small constant $\nu > 0$. Let e_1, \dots, e_m be the edges of H , and observe that the number of vertices contained in each edge is distributed as $\text{Bin}(n, p)$. Then the probability that e_i has an even number of vertices is

$$\begin{aligned} \sum_{j \text{ even}} \binom{n}{j} p^j (1-p)^{n-j} &= \frac{\sum_j \binom{n}{j} p^j (1-p)^{n-j} + \sum_j \binom{n}{j} (-p)^j (1-p)^{n-j}}{2} \\ &= \frac{1}{2}(1 + (1-2p)^n) = 1/2 + o(1), \end{aligned} \quad (6)$$

where we used the fact that $|1 - 2p|^n \leq \max\{(1 - 2\nu)^n, e^{-2pn}\} = o(1)$ as $n \rightarrow \infty$. Hence, the probability all the edges of H contain an even number of vertices is $(1/2 + o(1))^m = o(1)$. Therefore, *w.h.p.* H has an edge with an odd number of vertices, and thus has discrepancy at least 1. \square

Proposition 3.1 trivially implies Theorem 1.1 in the regime in which $2^{-n/m} \sqrt{pn} = O(1)$. We now use Lemma 2.4 to prove the remaining cases of Theorem 1.1 via a simple first moment argument:

Proof of Theorem 1.1. Suppose that $H = (V, E)$ is generated from $\mathbb{H}(n, m, p)$ with $pn \rightarrow \infty$, $pm \rightarrow \infty$ and p bounded away from 1, as $n \rightarrow \infty$. We define

$$\hat{f} = \hat{f}(n) = \begin{cases} 2^{-n/m} \sqrt{p(1-p)n} & \text{if } m = O(n), \\ \sqrt{p(1-p)n \log \gamma} & \text{if } m \gg n, \end{cases}$$

where $\gamma = \min\{pn, m/n\}$, and choose a sufficiently small constant $\kappa > 0$. To prove the theorem, it suffices to show that *w.h.p.* $\text{disc}(H) \geq \max\{\kappa \hat{f}, 1\}$. Note that this is trivially true when $\hat{f} \leq 1/\kappa$, in

view of Proposition 3.1. So we will assume that $\hat{f} > 1/\kappa$, and show that *w.h.p.* $\text{disc}(H) \geq \kappa\hat{f}$. Let Ψ be the set of all colourings $\psi : V \rightarrow \{-1, 1\}$, and let Z denote the number of colourings $\psi \in \Psi$ with discrepancy $\text{disc}(\psi) \leq \kappa\hat{f}$. In the rest of the proof, we show that $\mathbb{E}[Z] = o(1)$, which implies that *w.h.p.* $Z = 0$ and thus $\text{disc}(H) \geq \kappa\hat{f}$.

Since the random edges e_1, \dots, e_m of H are *i.i.d.*,

$$\mathbb{E}[Z] = \sum_{\psi \in \Psi} \mathbb{P}[\text{disc}(\psi) \leq \kappa\hat{f}] = \sum_{\psi \in \Psi} \mathbb{P}[|\psi(e_1)| \leq \kappa\hat{f}]^m. \quad (7)$$

Note that $\psi(e_1) = \sum_{i=1}^n \psi(v_i) \mathbf{1}_{v_i \in e_1}$, where $\mathbf{1}_{v_i \in e_1}$ denotes the indicator random variable of the event that edge e_1 contains vertex v_i , so $\psi(e_1)$ is distributed as $S_{\mathbf{a}, p}$ in Section 2 (with $a_i = \psi(v_i)$ and $p_i = \mathbb{P}[v_i \in e_1]$). Hence, by applying (5) in Lemma 2.4 (with $\varepsilon = 0$, $\zeta = p$ and $[L, R] = [-\kappa\hat{f}, \kappa\hat{f}]$) to each one of the 2^n terms of the last sum in (7), it follows that

$$\mathbb{E}[Z] \leq 2^n \left(\frac{c_{uni} + 2\kappa\hat{f}/\sqrt{2\pi}}{\sqrt{p(1-p)n}} \right)^m \leq 2^n \left(\frac{\kappa\hat{f}(c_{uni} + \sqrt{2/\pi})}{\sqrt{p(1-p)n}} \right)^m,$$

where we also used that $\kappa\hat{f} > 1$. Let us consider first the case that $m = O(n)$. Then, from the definition of \hat{f} and choosing κ so that $\kappa < 1/(c_{uni} + \sqrt{2/\pi})$,

$$\mathbb{E}[Z] \leq \left(\kappa(c_{uni} + \sqrt{2/\pi}) \right)^m = o(1).$$

Now suppose that $m \gg n$. In this case, we bound the factor $\mathbb{P}[|\psi(e_1)| \leq \kappa\hat{f}]$ on the right-hand side of (7) using the tighter inequality (4) in Lemma 2.4 instead of (5). Then,

$$\mathbb{E}[Z] \leq 2^n \left(\frac{c_{uni}}{\sqrt{p(1-p)n}} + \left(1 - \exp\left(-\frac{(2\kappa\hat{f})^2}{2\pi p(1-p)n}\right) \right)^{1/2} \right)^m.$$

Recall that here we have $\hat{f} = \sqrt{p(1-p)n \log \gamma}$, since $m \gg n$. Then, setting $C := 2\kappa^2/\pi$ and choosing κ so that $C < 1/2$, we may simplify:

$$\begin{aligned} \mathbb{E}[Z] &\leq 2^n \left(\frac{c_{uni}}{\sqrt{p(1-p)n}} + \left(1 - \gamma^{-2\kappa^2/\pi} \right)^{1/2} \right)^m \\ &= \left(2^{n/m} \right)^m \left(\frac{c_{uni}}{\sqrt{p(1-p)n}} + (1 - \gamma^{-C})^{1/2} \right)^m. \end{aligned}$$

Note that when $0 < n/m < 1$, we have $1 < 2^{n/m} < 1 + n/m$. Again using that $m \gg n$, for n sufficiently large we have $0 < n/m < 1$, so $2^{n/m} = 1 + O(n/m)$. Considering also that $\frac{c_{uni}}{\sqrt{p(1-p)n}} = O(1/\sqrt{pn})$, we obtain

$$\mathbb{E}[Z] = (1 + O(n/m))^m \left(O(1/\sqrt{pn}) + (1 - \gamma^{-C})^{1/2} \right)^m.$$

Now, recall that $\gamma = \min\{pn, m/n\}$, so $1/\sqrt{pn} = o(\gamma^{-C})$, $n/m = o(\gamma^{-C})$, and $m/\gamma^C \rightarrow \infty$. Then

$$\begin{aligned} \mathbb{E}[Z] &= \left(1 - \frac{1}{2}\gamma^{-C}(1 + o(1)) \right)^m \\ &= \exp\left(-\frac{m}{2}\gamma^{-C}(1 + o(1))\right) = o(1). \end{aligned}$$

In either case, $\mathbb{E}[Z] = o(1)$, and therefore *w.h.p.* $\text{disc}(H) \geq \kappa\hat{f}$. \square

3.2. The Edge-Dependent Model. In this section, we derive a lower bound on the discrepancy of a hypergraph generated from the edge-dependent model and prove Theorem 1.2. In view of our previous results for the edge-independent model, one natural approach is to compare both models $\mathcal{H}(n, m, d)$ and $\mathbb{H}(n, m, p)$ via a **coupling procedure**. For instance, let $m = n$ and $d \gg \log n$ for simplicity, and suppose that we can generate (H_1, H_2) with $H_1 \sim \mathcal{H}(n, n, d)$ and $H_2 \sim \mathbb{H}(n, n, p)$, with edge sets $E(H_1) = \{e_1^1, \dots, e_n^1\}$ and $E(H_2) = \{e_1^2, \dots, e_n^2\}$, in such a way that *w.h.p.* for every $i = 1, \dots, n$ we have $|e_i^1 \Delta e_i^2| \leq \eta$, for some suitable $\eta = \eta(n)$ (where Δ denotes the symmetric difference). In particular, this implies that *w.h.p.* $|\text{disc}(H_1) - \text{disc}(H_2)| \leq \eta$, and thus $\text{disc}(H_1) = \Omega(\sqrt{d} + \eta)$ by Theorem 1.1. Unfortunately, since the standard deviation of the size of an edge is $\Theta(\sqrt{d})$ in either model, most *naïve* attempts to build such a coupling require $\eta \gg \sqrt{d}$, which is too large for our purposes. (In fact, it is not hard to build such a coupling with any $\eta \gg \sqrt{d \log n}$.) As a result, while it is conceivable that a more delicate coupling argument works, we abandon this approach. Instead, we handle the dependencies of the edges of H_2 by applying a careful conditioning argument, while generalizing how we apply Lemma 2.4.

As in the edge-independent model, we first prove a constant lower bound on the discrepancy of $H \sim \mathcal{H}(n, m, d)$.

Proposition 3.2. *Suppose $H \sim \mathcal{H}(n, m, d)$ with $m \rightarrow \infty$, $dn/m \rightarrow \infty$ and d/m bounded away from 1 as $n \rightarrow \infty$. Then *w.h.p.* $\text{disc}(H) \geq 1$.*

Proof. By hypothesis, $p = d/m \leq 1 - \nu$ for some sufficiently small constant $\nu > 0$. Our goal is to show that *w.h.p.* there is some edge in H containing an odd number of vertices, so $\text{disc}(H) \geq 1$. As in the proof of Proposition 3.1, the probability that an edge e_i has an odd number of vertices is $1/2 + o(1)$ (see (6)). Therefore, the expected number W of edges with an odd number of vertices is $(1 + o(1))m/2$. Similarly, we have the following:

Claim 1. The probability that two different edges (say, e_1 and e_2) have an odd number of vertices is $1/4 + o(1)$.

In view of this claim, and summing over all $m(m-1)$ ordered pairs of different edges, we get that $\mathbb{E}W(W-1) \sim m^2/4$. This implies that $\mathbf{Var}W = \mathbb{E}W(W-1) + \mathbb{E}W - (\mathbb{E}W)^2 = o((\mathbb{E}W)^2)$ and, by a standard application of Chebyshev's inequality, we conclude that *w.h.p.* $W \sim m/2 \rightarrow \infty$. This proves the proposition under the assumption that Claim 1 holds.

To prove Claim 1, we first consider for each $i \in [n]$ the random vector $(\mathbf{1}_{v_i \in e_1}, \mathbf{1}_{v_i \in e_2})$, where $\mathbf{1}_{v_i \in e_j}$ denotes the indicator random variable of the event that edge e_j contains vertex v_i . In particular, the bivariate probability generating function of $(\mathbf{1}_{v_i \in e_1}, \mathbf{1}_{v_i \in e_2})$ is

$$f_i(x, y) = \frac{d(d-1)}{m(m-1)}xy + \frac{d(m-d)}{m(m-1)}(x+y) + \frac{(m-d)(m-d-1)}{m(m-1)},$$

where for each $s, t \in \{0, 1\}$ the coefficient

$$[x^s y^t]f_i(x, y)$$

denotes the probability that $\mathbf{1}_{v_i \in e_1} = s$ and $\mathbf{1}_{v_i \in e_2} = t$. For instance, $[xy]f_i(x, y)$ is simply the probability that $v_i \in e_1$ and $v_i \in e_2$ which equals $\frac{d(d-1)}{m(m-1)}$, etc. Now we define the random vector

$$D = (D_1, D_2) = \sum_{i=1}^n (\mathbf{1}_{v_i \in e_1}, \mathbf{1}_{v_i \in e_2}),$$

where D_j is the number of vertices in edge e_j , for $j = 1, 2$. Note that the terms in the sum above are independent, since they correspond to the contributions from different vertices. Therefore, the bivariate probability generating function $f(x, y)$ of D satisfies

$$f(x, y) = \prod_{i=1}^n f_i(x, y) = \left(\frac{d(d-1)}{m(m-1)}xy + \frac{d(m-d)}{m(m-1)}(x+y) + \frac{(m-d)(m-d-1)}{m(m-1)} \right)^n.$$

By definition of bivariate probability generating function, for $0 \leq i, j \leq n$, the coefficient

$$c_{ij} := [x^i y^j] f(x, y)$$

gives the probability that edge e_1 has i ones and edge e_2 has j ones. Hence,

$$f(1, 1) - f(-1, 1) - f(1, -1) + f(-1, -1) = \sum_{i, j \geq 0} (1 - (-1)^i - (-1)^j + (-1)^{i+j}) c_{ij} = 4 \sum_{i, j \text{ odd}} c_{ij},$$

so the probability that both edges have an odd number of vertices is

$$\frac{f(1, 1) - f(-1, 1) - f(1, -1) + f(-1, -1)}{4} = \frac{1 - 2 \left(1 - \frac{2d}{m}\right)^n + \left(1 - 4 \frac{d(m-d)}{m(m-1)}\right)^n}{4} = 1/4 + o(1).$$

The first step in the above equation follows from direct substitution and straightforward although tedious calculations. In the second step, we used the facts that $d/m \leq 1 - \nu$ and $dn/m \rightarrow \infty$ in order to bound the error term

$$\left| 2 \left(1 - \frac{2d}{m}\right)^n + \left(1 - 4 \frac{d(m-d)}{m(m-1)}\right)^n \right| \leq 2 \max\{(1 - 2\nu)^n, e^{-2dn/m}\} + e^{-4\nu dn/m} = o(1).$$

This proves Claim 1 and finishes the proof of the proposition. \square

Remark 6. It is conceivable that in the proof of the proposition above one could obtain an upper bound on $\mathbb{P}[W = 0]$ that is exponentially small in m , in the same spirit as in the proof of Proposition 3.1. However, this would require some additional work due to the fact that the edges of $H \sim \mathcal{H}(n, m, d)$ are *not* formed independently.

Proposition 3.2 trivially implies Theorem 1.2 in the regime in which $2^{-n/m} \sqrt{dn/m} = O(1)$. To prove the remaining cases, we will generalize the ideas we used in the proof of Theorem 1.1. However, the dependencies among the edges make the argument much more delicate.

Proof of Theorem 1.2. Suppose that $H = (V, E)$ is generated from $\mathcal{H}(n, m, d)$ with $m = m(n)$ and $d = d(n)$ satisfying $dn/m \rightarrow \infty$ and $d \rightarrow \infty$ (as $n \rightarrow \infty$) and with $p = d/m \leq c$ for some constant $0 < c < 1$. For short, we use \mathcal{H} to denote the sample space of the distribution, i.e. the set of all possible outcomes of H . Fix a constant $0 \leq \varepsilon < \min\{1, 1/c - 1\}$, and define

$$\hat{f} = \hat{f}(n) := \begin{cases} 2^{-n/m} \sqrt{pn(1-\varepsilon)(1-c(1+\varepsilon))} = \Theta(2^{-n/m} \sqrt{dn/m}) & \text{if } m = O(n), \\ \sqrt{pn(\log \gamma)(1-\varepsilon)(1-c(1+\varepsilon))} = \Theta(\sqrt{pn \log \gamma}) & \text{if } m \gg n, \end{cases}$$

where $\gamma = \min\{pn, m/n\}$. Let $\kappa > 0$ be a sufficiently small constant. To prove Theorem 1.2, it suffices to show that *w.h.p.* $\text{disc}(H) \geq \max\{\kappa \hat{f}, 1\}$. Note that this is trivially true when $\hat{f} \leq 1/\kappa$, in view of Proposition 3.2. So we will assume that $\hat{f} > 1/\kappa$, and show that *w.h.p.* $\text{disc}(H) \geq \kappa \hat{f}$. Note that this assumption ensures that $2^{-n/m} \sqrt{dn/m} = \Omega(1)$, and so after taking logarithms, we get that $n = O(m \log d)$. Thus, we may assume that there are not *too* many more vertices than edges.

Let Z be the number of colourings $\psi : V \rightarrow \{-1, 1\}$ with discrepancy $\text{disc}(\psi) \leq \kappa \hat{f}$. We would like to prove an analogue of (7) in order to bound $\mathbb{E}Z$, but unfortunately the random edges e_1, \dots, e_m of H are no longer *i.i.d.* In order to overcome this obstacle, we introduce some random variables that will play an essential role in the analysis of Z . For each $j = 1, \dots, m$ and $k = 1, \dots, n$, let $A_{j,k} := \mathbf{1}_{[v_k \in e_j]}$ be the $\{0, 1\}$ value of the (j, k) entry of the incidence matrix \mathbf{A} of H , and let $\mathbf{A}_j = (A_{j,1}, \dots, A_{j,n})$ denote the j -th row of \mathbf{A} . Moreover, for each $k = 1, \dots, n$, we define

$$B_{0,k} := d \quad \text{and} \quad B_{i,k} := d - \sum_{j=1}^i A_{j,k} \quad \text{for } i = 1, \dots, m. \quad (8)$$

In other words, $B_{i,k}$ counts the number of ones that appear in the k -th column of \mathbf{A} below the i -th row (recall that each column of A has exactly d ones). Note that each $B_{i,k}$ can be expressed as a function of the first i rows of \mathbf{A} , and moreover the distribution of $A_{i+1,k}$ conditional on the outcome of $A_{1,k}, \dots, A_{i,k}$ can be described solely in terms of $B_{i,k}$. More formally, for each $i = 1, \dots, m$, let \mathcal{F}_i be the sigma algebra generated by $\mathbf{A}_1, \dots, \mathbf{A}_i$, and let $\mathcal{F}_0 := \{\emptyset, \mathcal{H}\}$ denote the trivial sigma algebra. Then, for each $i = 0, \dots, m$ and $k = 1, \dots, n$, the random variable $B_{i,k}$ is measurable with respect to \mathcal{F}_i , and (for $i < m$)

$$P_{i,k} := \mathbb{P}[A_{i+1,k} = 1 \mid \mathcal{F}_i] = \frac{B_{i,k}}{m-i}.$$

Intuitively speaking, we would like that the above conditional probabilities remain close to p (on average) as we reveal new rows of \mathbf{A} , at least for a large number of rows. However, this is not possible for certain parameter ranges, such as when d is small and concentration guarantees break down. For instance, if $n = m$ and $d = o(\log n)$, then *w.h.p.* there will exist a column j with $P_{j,n/2} \geq 2p$. Fortunately, there is a less restrictive event we can impose which suffices for our application of Lemma 2.4. For each $i = 0, \dots, m-1$, we define Q_i to be the event that for every $0 \leq j \leq i$

$$\sum_{k=1}^n P_{j,k} \geq (1-\varepsilon)pn \quad \text{and} \quad P_{j,k} \leq (1+\varepsilon)c \quad \text{for } k = 1, \dots, n.$$

Here $(1+\varepsilon)c < 1$ from our choice of ε , and so the second condition of Q_i imposes that the conditional probabilities are bounded away from 1. Observe that $\mathcal{H} = Q_0 \supseteq \dots \supseteq Q_{m-1}$ is a decreasing sequence of events, and each Q_i is \mathcal{F}_i -measurable by construction. Let $\alpha := \max\{n/(n+m), 1/2\}$. We need the following technical result, which we prove in Section 3.3.

Proposition 3.3. *Under the assumptions in the proof of Theorem 1.2, event $Q_{\lfloor \alpha m \rfloor}$ holds *w.h.p.**

Now let Ψ be the set of all colourings $\psi : V \rightarrow \{-1, 1\}$, and pick an arbitrary $\psi \in \Psi$. For each $i = 1, \dots, m$, let R_i^ψ denote the event that $|\psi(e_i)| \leq \kappa \hat{f}$, and let $R_{\leq i}^\psi := \bigcap_{j=1}^i R_j^\psi$. (By convention, $R_{\leq 0}^\psi = \mathcal{H}$.) Clearly, R_i^ψ and $R_{\leq i}^\psi$ are \mathcal{F}_i -measurable. Note that, conditional upon any outcome of $\mathbf{A}_1, \dots, \mathbf{A}_{i-1}$ satisfying Q_{i-1} , the random variable $\psi(e_i)$ is distributed as $S_{\mathbf{a}, \mathbf{p}}$ in Section 2 (with $a_k = \psi(v_k)$ and $p_k = \mathbb{P}[v_k \in e_i]$) and it satisfies the conditions of Lemma 2.4 (with $\zeta = (1+\varepsilon)c < 1$ and $[L, R] = [-\kappa \hat{f}, \kappa \hat{f}]$). Hence, we can use that lemma to bound the conditional probability of R_i^ψ . We first consider the sparse regime of $m = O(n)$. Recall that, in this regime, $\hat{f} = 2^{-n/m} \sqrt{pn(1-\varepsilon)(1-c(1+\varepsilon))}$. By Lemma 2.4, choosing κ so that $\kappa < 1 / \left(3(c_{uni} + \sqrt{2/\pi})\right)$ and since $\kappa \hat{f} > 1$,

$$\mathbb{P}[R_i^\psi \mid \mathcal{F}_{i-1}] \mathbf{1}_{Q_{i-1}} \leq \frac{c_{uni} + 2\kappa \hat{f} / \sqrt{2\pi}}{\sqrt{(1-\zeta)(1-\varepsilon)pn}} \leq \frac{\kappa \hat{f} (c_{uni} + \sqrt{2/\pi})}{\sqrt{(1-\zeta)(1-\varepsilon)pn}} < 2^{-n/m} / 3. \quad (9)$$

In particular, since $R_{\leq i-1}^\psi \cap Q_{i-1}$ is \mathcal{F}_{i-1} -measurable and is contained in Q_{i-1} ,

$$\mathbb{P}[R_1^\psi] \leq 2^{-n/m} / 3 \quad \text{and} \quad \mathbb{P}[R_i^\psi \mid R_{\leq i-1}^\psi \cap Q_{i-1}] \leq 2^{-n/m} / 3 \quad \text{for } i = 2, \dots, m.$$

Thus, for each $i = 2, \dots, m$,

$$\mathbb{P}[R_{\leq i}^\psi \cap Q_{i-1}] = \mathbb{P}[R_i^\psi \mid R_{\leq i-1}^\psi \cap Q_{i-1}] \cdot \mathbb{P}[R_{\leq i-1}^\psi \cap Q_{i-1}] \leq \left(2^{-n/m} / 3\right) \mathbb{P}[R_{\leq i-1}^\psi \cap Q_{i-1}],$$

and inductively

$$\mathbb{P}[R_{\leq i}^\psi \cap Q_{i-1}] \leq \left(2^{-n/m} / 3\right)^i.$$

Let $t := \lceil \alpha m \rceil$. Next, we will bound $\text{disc}(H)$ from below based on the discrepancies of the first t rows of \mathbf{A} when Q_{t-1} holds. First note that, since $t \geq nm/(n+m)$,

$$\mathbb{P}[R_{\leq t}^\psi \cap Q_{t-1}] \leq \left(2^{-n/m}/3\right)^t \leq \left(2^{-n/m}/3\right)^{nm/(n+m)} = 2^{-n}(2/3)^{nm/(n+m)} = o(2^{-n}), \quad (10)$$

and then, by applying Markov's inequality to the random variable $Z\mathbf{1}_{Q_{t-1}}$,

$$\mathbb{P}[\text{disc}(H) \leq \kappa \hat{f} \text{ and } Q_{t-1}] \leq \mathbb{E}[Z\mathbf{1}_{Q_{t-1}}] = \sum_{\psi \in \Psi} \mathbb{P}[R_{\leq m}^\psi \cap Q_{t-1}] \leq \sum_{\psi \in \Psi} \mathbb{P}[R_{\leq t}^\psi \cap Q_{t-1}] = o(1). \quad (11)$$

Next, we consider the dense regime of $m \gg n$. In that case, we obtain an analogue of (9) by using the tighter inequality (4) in Lemma 2.4 instead of (5). Here, we have $\hat{f} = \sqrt{pn \log \gamma(1-\varepsilon)(1-c(1+\varepsilon))}$. With $\zeta = (1+\varepsilon)c < 1$ and choosing the constant κ to satisfy $C := 2\kappa^2/\pi < 1/2$, we get

$$\begin{aligned} \mathbb{P}[R_i^\psi \mid \mathcal{F}_{i-1}] \mathbf{1}_{[Q_{i-1}]} &\leq \frac{c_{\text{uni}}}{\sqrt{(1-\zeta)(1-\varepsilon)pn}} + \left(1 - \exp\left(-\frac{(2\kappa\hat{f})^2}{2\pi(1-\zeta)(1-\varepsilon)pn}\right)\right)^{1/2} \\ &= O(1/\sqrt{pn}) + \left(1 - \gamma^{-2\kappa^2/\pi}\right)^{1/2} = 1 - \Theta(\gamma^{-C}), \end{aligned}$$

where we used the fact that $1/\sqrt{pn} = o(\gamma^{-C})$. Reasoning as before, we obtain the following analogue of (10):

$$\mathbb{P}[R_{\leq t}^\psi \cap Q_{t-1}] \leq (1 - \Theta(\gamma^{-C}))^t \leq (1 - \Theta(\gamma^{-C}))^{m/2} = e^{-n\Theta(\gamma^{-C}m/n)} = o(2^{-n}),$$

where we used the facts that $t \geq m/2$ and $n/m = o(\gamma^{-C})$. As a result, our bound in (11) is also valid when $m \gg n$ as well. Then, in either regime ($m = O(n)$ or $m \gg n$),

$$\mathbb{P}[\text{disc}(H) \leq \kappa \hat{f}] \leq \mathbb{P}[\text{disc}(H) \leq \kappa \hat{f} \text{ and } Q_{t-1}] + \mathbb{P}[\neg Q_{t-1}] = o(1), \quad (12)$$

by (11), Proposition 3.3 and the fact $Q_{\lceil \alpha m \rceil - 1} \supseteq Q_{\lfloor \alpha m \rfloor}$. This shows that *w.h.p.* $\text{disc}(H) \geq \kappa \hat{f}$, and concludes the proof of Theorem 1.2. \square

3.3. Proof of Proposition 3.3. In this section, we prove Proposition 3.3. For any $m \in \mathbb{N}$, let $[m] := \{1, 2, \dots, m\}$ and $[0] := \emptyset$. Suppose that $J \subseteq [m]$ is a fixed subset of size $0 \leq j \leq m$. If $S \subseteq [m]$ is a random subset of size d , then the distribution of the random variable $|S \cap J|$ is said to be **hypergeometric** with parameters m, d and j . We denote this distribution by $\text{Hyper}(m, d, j)$ in what follows. Now, $\text{Hyper}(m, d, j)$ is at least as concentrated about its expectation as the binomial distribution, $\text{Bin}(d, j/m)$ (see Chapter 21 in [15] for details). As such, standard Chernoff bounds ensure the following:

Theorem 3.4. *Suppose that $X \sim \text{Hyper}(m, d, j)$, and $\mu := \mathbb{E}[X] = dj/m$. In this case, for every $0 < \lambda < 1$,*

$$\mathbb{P}(|X - \mu| \geq \lambda\mu) \leq 2 \exp\left(\frac{-\lambda^2\mu}{3}\right).$$

Let \mathbf{A} be the adjacency matrix of $H \sim \mathcal{H}(n, m, d)$, which has exactly d ones in each column at random positions. Let $p = d/m$. Recall that, for each $i = 0, \dots, m$ and $k = 1, \dots, n$, the random variable $B_{i,k}$ counts the number of ones in the k -th column and below the i -th row of \mathbf{A} (cf. (8)). Also recall $P_{i,k} := B_{i,k}/(m-i)$ (for $i < m$). Clearly, $B_{i,k} \sim \text{Hyper}(m, d, m-i)$, so we may apply Theorem 3.4 to control the value of $B_{i,k}$, and thus of $P_{i,k}$. Let $\alpha := \max\{n/(n+m), 1/2\}$ and $t := \lceil \alpha m \rceil$. For a fixed column k , our goal is to show that *w.h.p.* $B_{i,k}$ remains “close” to $\mathbb{E}[B_{i,k}] = d(m-i)/m$ for all $i = 1, \dots, t$. By combining the error term in Theorem 3.4 with a naïve union bound, we can bound the probability of failure by something of the order of $m \exp(\Theta(-d))$, which does not tend to 0 unless $d = \Omega(\log m)$. To overcome this, we need a more subtle argument in which we take the union bound over a smaller set of indices i and take into account that $B_{i,k}$

does not change too much between two consecutive values of i . This is made more precise in the following claim:

Proposition 3.5. *Assume $0 < \alpha, \lambda, \xi < 1$ with $\xi \geq 1/m$ and $\alpha + \xi < 1$. Fix $1 \leq k \leq n$. Then, with probability at least*

$$1 - 8\xi^{-1} \exp\left(\frac{-d\lambda^2(1 - \alpha - \xi)^2}{3}\right),$$

it holds that, for all $i = 0, \dots, \lfloor \alpha m \rfloor$,

$$(1 - \lambda) \left(1 + \frac{\xi}{1 - \alpha - \xi}\right)^{-1} p \leq P_{i,k} \leq (1 + \lambda) \left(1 + \frac{\xi}{1 - \alpha - \xi}\right) p. \quad (13)$$

Proof. As the columns of \mathbf{A} are identically distributed, we may assume that $k = 1$ in what follows. We thus drop the index k from the notation of $A_{i,k}, B_{i,k}, P_{i,k}$ for simplicity. Recall $B_i \sim \text{Hyper}(m, d, m - i)$ with $\mathbb{E}B_i = p(m - i)$ for each $i = 0, \dots, m$.

Our goal is to show that $B_0, \dots, B_{\lfloor \alpha m \rfloor}$ (and thus also $P_0, \dots, P_{\lfloor \alpha m \rfloor}$) are concentrated around their mean. The key idea behind the argument is to first obtain concentration for a smaller subset of random variables, and then extend that to the rest. Here is a high-level description of our strategy in simple terms. First, we will partition the set of indices $[\lfloor \alpha m \rfloor]$ into approximately α/ξ intervals of length $\lfloor \xi m \rfloor$. Then, we will prove that for every i at the upper end of an interval B_i is concentrated, by applying Theorem 3.4 and taking the union bound over all intervals. Finally, we will show that B_i cannot change too much for i within each individual interval. The advantage of this approach is that the number of terms in the union bound is around α/ξ , which can be made much smaller than αm by taking ξ sufficiently larger than $1/m$.

Let $r := \lceil (m - 1)/\lfloor \xi m \rfloor \rceil$, which satisfies $1 \leq r \leq m - 1$ by assumption. Our first goal is to partition the set $[m - 1]$ into r intervals, each of size at most ξm . For each $q = 0, \dots, r - 1$ let $I_q := [q\lfloor \xi m \rfloor]$, and let $I_r := [m - 1]$. Then, setting $\tilde{I}_q := I_q \setminus I_{q-1}$ for $q = 1, \dots, r$, gives us the desired partition $\tilde{I}_1, \dots, \tilde{I}_r$ of $[m - 1]$. Now, let $r_0 := \lceil \lfloor \alpha m \rfloor / \lfloor \xi m \rfloor \rceil$. Since $r_0 \lfloor \xi m \rfloor \geq \lfloor \alpha m \rfloor$, the set $[\lfloor \alpha m \rfloor]$ is contained in $\bigcup_{q=1}^{r_0} \tilde{I}_q$. We claim that $r_0 \leq r - 1$. Clearly, $r_0 \leq r$ since $\lfloor \alpha m \rfloor \leq m - 1$. If $r_0 = r$, then $(m - 1) - \lfloor \alpha m \rfloor < \lfloor \xi m \rfloor$, which implies $\lfloor \alpha m \rfloor + \lfloor \xi m \rfloor \geq m$ by integrality. This contradicts the fact that $\lfloor \alpha m \rfloor + \lfloor \xi m \rfloor \leq (\alpha + \xi)m < m$. As a result, $r_0 \leq r - 1$, as claimed.

For each $0 \leq q \leq r - 1$, define $Y_q := B_{q\lfloor \xi m \rfloor}$ and let $Y_r := B_{m-1}$. In other words, each Y_q counts the number of ones in the k -th column of \mathbf{A} below all the rows indexed by I_q . We will prove that the variables Y_0, \dots, Y_{r_0} are concentrated around their mean, and from that derive a concentration result for $B_0, \dots, B_{\lfloor \alpha m \rfloor}$. Observe that Y_r must be defined slightly differently, due to divisibility issues. Fortunately, our argument will only require the analysis of Y_0, \dots, Y_{r_0} , where $r_0 \leq r - 1$, so this fact will cause no trouble. For $q = 0, \dots, r - 1$,

$$\mathbb{E}Y_q = p(m - |I_q|) = p(m - q\lfloor \xi m \rfloor),$$

and therefore, for every $q = 1, \dots, r_0$,

$$\frac{\mathbb{E}Y_{q-1}}{\mathbb{E}Y_q} = 1 + \frac{\lfloor \xi m \rfloor}{m - q\lfloor \xi m \rfloor} \leq 1 + \frac{\lfloor \xi m \rfloor}{m - \lfloor \alpha m \rfloor - \lfloor \xi m \rfloor} \leq 1 + \frac{\xi}{1 - \alpha - \xi}, \quad (14)$$

where we also used the fact that $r_0 \lfloor \xi m \rfloor \leq \lfloor \alpha m \rfloor + \lfloor \xi m \rfloor$. Now, let E be the event that

$$|Y_q - \mathbb{E}Y_q| \leq \lambda \mathbb{E}Y_q \quad \text{for all } q = 0, \dots, r_0.$$

A direct application of Theorem 3.4 yields

$$\mathbb{P}(\neg E) \leq \sum_{q=0}^{r_0} 2 \exp\left(\frac{-\lambda^2 p^2 (m - q\lfloor \xi m \rfloor)^2}{3d}\right) \leq 2(r_0 + 1) \exp\left(\frac{-\lambda^2 p^2 (m - r_0 \lfloor \xi m \rfloor)^2}{3d}\right).$$

Using the fact that $r_0 \lfloor \xi m \rfloor \leq (\alpha + \xi)m$ and the rough bound $r_0 + 1 \leq \frac{\alpha m}{\xi m/2} + 2 \leq \frac{4}{\xi}$, we conclude that

$$\mathbb{P}(\neg E) \leq (8/\xi) \exp\left(\frac{-d\lambda^2(1-\alpha-\xi)^2}{3}\right). \quad (15)$$

Finally, we turn our attention to $B_1, \dots, B_{\lfloor \alpha m \rfloor}$. For each $i \in [\lfloor \alpha m \rfloor]$, we pick $q \in [r_0]$ such that $i \in \tilde{I}_q$. Then, by monotonicity,

$$Y_q \leq B_i \leq Y_{q-1} \quad \text{and} \quad \mathbb{E}Y_q \leq \mathbb{E}B_i \leq \mathbb{E}Y_{q-1}.$$

Combining this with (14) yields

$$\mathbb{E}Y_{q-1} \leq \left(1 + \frac{\xi}{1-\alpha-\xi}\right) \mathbb{E}B_i \quad \text{and} \quad \mathbb{E}Y_q \geq \left(1 + \frac{\xi}{1-\alpha-\xi}\right)^{-1} \mathbb{E}B_i.$$

As a result, event E implies that for every $1 \leq i \leq \lfloor \alpha m \rfloor$,

$$(1-\lambda)\mathbb{E}Y_q \leq Y_q \leq B_i \leq Y_{q-1} \leq (1+\lambda)\mathbb{E}Y_{q-1},$$

and hence

$$(1-\lambda) \left(1 + \frac{\xi}{1-\alpha-\xi}\right)^{-1} \mathbb{E}B_i \leq B_i \leq (1+\lambda) \left(1 + \frac{\xi}{1-\alpha-\xi}\right) \mathbb{E}B_i. \quad (16)$$

(Note that the equation above is also valid for $i = 0$, since $B_0 = d = \mathbb{E}B_0$.) Dividing (16) by $m - i$, we conclude that event E implies that, for every $0 \leq i \leq \lfloor \alpha m \rfloor$,

$$(1-\lambda) \left(1 + \frac{\xi}{1-\alpha-\xi}\right)^{-1} p \leq P_i \leq (1+\lambda) \left(1 + \frac{\xi}{1-\alpha-\xi}\right) p.$$

Our bound on $\mathbb{P}(\neg E)$ in (15) completes the proof of the proposition. \square

Corollary 3.6. *Suppose $m \geq d \rightarrow \infty$ and $n = O(m \log d)$ as $n \rightarrow \infty$. Set $p = d/m$ and $\alpha = \max\{n/(n+m), 1/2\}$. Given any fixed constant $0 < \varepsilon < 1$ and any $1 \leq k \leq n$, the following holds with probability at least $1 - \exp(-\Omega(d/\log^3 d))$. For every $i = 0, \dots, \lfloor \alpha m \rfloor$,*

$$|P_{i,k} - p| \leq \varepsilon p. \quad (17)$$

Proof. Since the probability bound in the statement is asymptotic as $n \rightarrow \infty$, we will implicitly assume throughout the proof that n is sufficiently large for all the inequalities therein to be valid. First, define $\lambda := \xi := 1/\log^{3/2} d$. Clearly, $\xi \geq 1/d \geq 1/m$. Observe that, since $n = O(m \log d)$, we have

$$1 - \alpha = \min\{m/(n+m), 1/2\} = \Omega(1/\log d). \quad (18)$$

In particular $\alpha + \xi < 1$, and thus all the assumptions in Proposition 3.5 are satisfied. Moreover,

$$\frac{\xi}{1-\alpha-\xi} = O\left(\frac{1/\log^{3/2} d}{1/\log d}\right) = O(\log^{-1/2} d).$$

As a result, we can relax the inequalities in (13) to

$$P_{i,k} = \left(1 + O(\log^{-3/2} d)\right) \left(1 + O(\log^{-1/2} d)\right) p = (1 + o(1))p,$$

which implies that $|P_{i,k} - p| \leq \varepsilon p$ (eventually for n sufficiently large). In view of Proposition 3.5, this fails for some $i = 0, \dots, \lfloor \alpha m \rfloor$ with probability at most

$$\begin{aligned} 8\xi^{-1} \exp\left(\frac{-d\lambda^2(1-\alpha-\xi)^2}{3}\right) &\leq 8(\log^{3/2} d) \exp\left(\frac{-d(1/2 - \log^{-3/2} d)^2}{3 \log^3 d}\right) \\ &= \exp(-\Omega(d/\log^3 d)). \end{aligned}$$

This finishes the proof of the corollary. \square

Now we are ready to prove Proposition 3.3, which we restate below in a more explicit form for convenience.

Proposition 3.7. *Let $0 < \varepsilon, c < 1$ be fixed constants with $(1 + \varepsilon)c < 1$. Assume that $d \rightarrow \infty$, $dn/m \rightarrow \infty$ and $n = O(m \log d)$ as $n \rightarrow \infty$, and suppose that $p = d/m \leq c$. Let $\alpha = \max\{n/(n + m), 1/2\}$. Then w.h.p., for every $i = 0, \dots, \lfloor \alpha m \rfloor$,*

$$\sum_{k=1}^n P_{i,k} \geq (1 - \varepsilon)pn, \quad (19)$$

and

$$P_{i,k} \leq (1 + \varepsilon)c \quad \text{for } k = 1, \dots, n. \quad (20)$$

Proof. For $k = 1, \dots, n$, we say that column k of \mathbf{A} is *controllable* if, for every $i = 0, \dots, \lfloor \alpha m \rfloor$, it holds that

$$|P_{i,k} - p| \leq \varepsilon_0 p,$$

where $\varepsilon_0 := \varepsilon/2$. Let $U \subseteq [n]$ denote the set of indices of the *uncontrollable* columns. Then, by Corollary 3.6 (with ε replaced by ε_0),

$$\mathbb{E}|U| \leq n \exp(-\Omega(d/\log^3 d)) = o(n).$$

Hence, we can apply Markov's inequality to ensure that $|U|/n \leq \varepsilon_0$ w.h.p. On the other hand, by applying the trivial lower bound to $P_{i,k}$ for each controllable column $k \in [n]$,

$$\sum_{k=1}^n P_{i,k} \geq (n - |U|)(1 - \varepsilon_0)p \geq (1 - \varepsilon_0)^2 pn \geq (1 - 2\varepsilon_0)pn,$$

w.h.p., thereby proving (19) (as $2\varepsilon_0 = \varepsilon$).

In order to verify that (20) holds, we first consider the regime in which $d \leq \log^2 n$. Observe then that *deterministically*

$$P_{i,k} \leq \frac{d}{m - i} \leq \frac{d}{(1 - \alpha)m},$$

for each $i = 1, \dots, \lfloor \alpha m \rfloor$ and $k = 1, \dots, n$. In particular, since $1 - \alpha = \Omega(\log^{-1} d)$ (in view of (18)) and $n = O(m \log d)$, it holds that

$$P_{i,k} = O(d(\log^2 d)/n) = o(1).$$

Thus, (20) holds in this regime, as $c(1 + \varepsilon) > 0$ is a fixed constant. On the other hand, if $d \geq \log^2 n$, we can apply Corollary 3.6 again, which ensures that with probability at least

$$1 - n \exp(-\Omega(d/\log^3 d)) = 1 - o(1)$$

we have

$$P_{i,k} \leq (1 + \varepsilon)p \leq (1 + \varepsilon)c$$

for all $i = 0, \dots, \lfloor \alpha m \rfloor$ and $k = 1, \dots, n$. The proof is therefore complete. \square

4. UPPER BOUNDING DISCREPANCY—PROOF OF THEOREM 1.3

The main tool we make use of is the algorithmic partial colouring lemma [19], as done in [6, 22, 23]. For convenience, we restate this lemma in the relevant hypergraph terminology, where we define a **fractional colouring** to be a relaxation of a (hypergraph) colouring, whose values are allowed to lie in the interval $[-1, 1]$:

Lemma 4.1 (Partial Colouring Lemma [19]). *Suppose that $H = (V, E)$ is a hypergraph with m edges and n vertices which are coloured by some fractional colouring $\rho : V \rightarrow [-1, 1]$. Moreover, assume that $\delta > 0$ and $(\lambda_e)_{e \in E}$ are non-negative values such that*

$$\sum_{e \in E} \exp(-\lambda_e^2/16) \leq \frac{n}{16}. \quad (21)$$

Under these assumptions, there exists some fractional colouring $\psi : V \rightarrow [-1, 1]$ for which

- (1) $|\psi(e) - \rho(e)| \leq \lambda_e |e|^{1/2}$ for all $e \in E$, and
- (2) $|\psi(v)| \geq 1 - \delta$ for at least $n/2$ vertices of V .

Moreover, ψ can be computed by a randomized algorithm in expected time

$$O((n + m)^3 \delta^{-2} \log(mn/\delta)).$$

Remark 7. As previously mentioned, Levy, Ramadas, and Rothvoss [18] showed that the algorithmic guarantee of Lemma 4.1 can be achieved using a deterministic algorithm (albeit with a worse run-time).

In what follows, it will be convenient to refer to ρ as the **target (fractional) colouring** and δ as the **rounding parameter**. We make use of Lemma 4.1 in the same way as in [6, 19, 22, 23]. In fact, we analyze the same *two-phase* algorithm considered by both Bansal and Meka [6] and Potukuchi [22, 23], though we must tune the relevant asymptotics carefully in order to achieve the bound claimed in Theorem 1.3. In particular, we shorten phase one and change the target discrepancy in each application of Lemma 4.1. These modifications allow us to derive more precise asymptotics in the studied range of parameters.

Let us suppose that $H = (V, E)$ is a hypergraph drawn from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$, where $p = d/m$. From now on, we assume that n is a power of 2 for convenience. This follows *w.l.o.g.* as we can always add extra vertices to V which do not lie in any of the edges of G . Given the lower bounds of Theorems 1.1 and 1.2, ideally we would like to compute an **output colouring**, $\phi : V \rightarrow \{-1, 1\}$ with matching discrepancy. However, without finer proof techniques, this does not seem fully attainable for the full parameter range of $m \gg n$. Let us fix $\mu := dn/m$. We recall the definition of $\beta = \beta(n)$ as given in the statement of Theorem 1.3: For all n sufficiently large, $\beta(n) \geq 1$ and

$$\beta(n)\mu \geq \log(m/n) (\log \mu + 2)^5. \quad (22)$$

Let $\hat{f} := \sqrt{\mu \log(m/n) \beta}$ be the target upper bound on the discrepancy that we are aiming to prove. Our argument is based on analyzing the ITERATED-COLOURING-ALGORITHM, which we now describe. Fix $t_1 := \lg \mu$, $0 \leq i \leq t_1$, and let $H_0 := H$, and $\delta := 1/n$. For convenience, we refer to the below procedure as **round i** . We define $\hat{f}_i := \hat{f}(i+2)^{-2}$ as the desired discrepancy bound to be attained in round i .

- (1) Remove all edges of $H_i = (V_i, E_i)$ of size less than or equal to \hat{f} .
- (2) Update λ_e to be $\hat{f}_i/|e|^{1/2}$ for each $e \in E_i$. Update the target colouring, which we denote by ρ_i , to be the previously computed colouring $\psi_{i-1} : V_{i-1} \rightarrow [-1, 1]$ restricted to V_i (here ψ_{-1} is the identically zero function by convention).
- (3) If $\sum_{e \in E_i} \exp(-\lambda_e^2/16) \leq |V_i|/16$, then apply Lemma 4.1 to H_i with the above values, yielding the fractional colouring $\psi_i : V_i \rightarrow [-1, 1]$. Otherwise, abort the round and declare an error.
- (4) Assuming an error did not occur, compute $S_i \subseteq V_i$, such that $|\psi_i(v)| \geq 1 - 1/n$ for all $v \in S_i$, where $|S_i| := |V_i|/2$. Afterwards, construct $H_{i+1} = (V_{i+1}, E_{i+1})$ by restricting the edges of E_i to V_{i+1} , where $V_{i+1} := V_i \setminus S_i$.

We refer to the rounds $i = 0, \dots, t_1$ as **phase one** of the algorithm's execution. Note that we refer to the vertices of S_i as **inactive** after round i , as the value ϕ assigns to them will not change at any point onwards. We refer to the remaining vertices as being **active**. Observe then the following proposition:

Proposition 4.2. *For each $0 \leq i \leq t_1$ and $e \in E_i$, it holds that*

$$|\psi_i(e) - \rho_i(e)| \leq \frac{\hat{f}}{(i+2)^2}.$$

Assuming that none of the rounds yielded an error, there are exactly $2^{-t_1}n = n\mu^{-1} = m/d$ active vertices at the end of phase one, as $t_1 = \lg \mu$. Heuristically, this means that we expect H_{t_1+1} to have edges of roughly constant size since $p = d/m$. As such, we can easily complete the colouring ϕ by executing the **phase two** procedure for rounds $t_1 + 1 \leq i \leq t_2$, where $t_2 := \lg(10n/\hat{f}) + 1$. The phase two procedure is identical to that of phase one, with the exception that in step 2, we update λ_e to be 0 (rather than $\hat{f}_i/|e|^{1/2}$) for each $e \in E_i$. Analogously, we observe the following.

Proposition 4.3. *For each $t_1 + 1 \leq i \leq t_2$ and $e \in E_i$, it holds that*

$$|\psi_i(e) - \rho_i(e)| = 0.$$

Assuming that none of the rounds yielded an error, there are exactly $2^{-t_2}n = n\hat{f}/20n = \hat{f}/20$ active vertices at the end of phase two. In order to complete the construction of ϕ , we conclude with a **post-processing phase**. That is, we arbitrarily assign -1 or 1 to any of the vertices which remain active at the end of phase two. Finally, we round each remaining fractional value assigned by ϕ to the nearest integer within $\{-1, 1\}$.

Let us assume that the above procedure succeeds in its execution on H ; that is, it does *not* abort during any iteration in either phase one or two. In this case, we conclude the proof by showing the next lemma.

Lemma 4.4. *If neither phase one nor phase two fails then Theorem 1.3 holds.*

Proof. For each $0 \leq i \leq t_2$, let us formally extend ψ_i to all of V . That is, define $\psi_i(v) := 0$ for each $v \in V \setminus V_i$, and keep ψ_i unchanged on V_i . Moreover, do the same for the target colouring ρ_i . Observe then that once phase two ends, ϕ can be expressed as a sum of differences involving the partial colourings $(\psi_i)_{i=0}^{t_2}$ and $(\rho_i)_{i=0}^{t_2-1}$. Specifically,

$$\phi(v) = \sum_{i=0}^{t_2} (\psi_i(v) - \rho_i(v)).$$

Let t_e be the time when edge e becomes smaller than \hat{f} or $t_e = t_2$ if it never happens. After applying Propositions 4.2 and 4.3 we get that

$$\begin{aligned} |\phi(e)| &\leq \hat{f} + \sum_{i=0}^{t_e} |\psi_i(e) - \rho_i(e)| \\ &\leq \hat{f} + \sum_{i=0}^{t_1} |\psi_i(e) - \rho_i(e)| + \sum_{i=t_1+1}^{t_2} |\psi_i(e) - \rho_i(e)| \\ &\leq \hat{f} + \sum_{i=0}^{\infty} \frac{\hat{f}}{(i+2)^2} = O(\hat{f}). \end{aligned}$$

The post-processing phase cannot increase the discrepancy that ϕ attains on any edge of E by more than \hat{f} for the remaining active vertices; as we already observed, there are at most $\hat{f}/20$ of them. The rounding of inactive vertices increases the discrepancy by at most 1. \square

Bounding the Failure Probability. First, we recall the following lemma proven in [6] by Bansal and Meka:

Lemma 4.5 (Lemma 6 in [6]). *Suppose that H is generated from $\mathcal{H}(n, m, d)$ and \mathbf{M} is a fixed $r \times \ell$ sub-matrix of the $m \times \ell$ incidence matrix \mathbf{A} of H . If $s \geq 10d\ell/m$, and $B(r, \ell, s)$ corresponds to the event in which each row of \mathbf{M} has at least s 1's, then*

$$\mathbb{P}[B(r, \ell, s)] \leq \exp\left(-\frac{rs \log((sm)/(d\ell))}{2}\right).$$

While this lemma is stated for the case when H is generated from $\mathcal{H}(n, m, d)$, the upper bound on $\mathbb{P}[B(r, \ell, s)]$ is proven by instead bounding the probability of the analogous event when H is generated from $\mathbb{H}(n, m, p)$ for $p = d/m$. As such, this lemma extends to the edge independent model. We now restate it in a form which will be more convenient for our purposes.

Lemma 4.6. *Suppose that H is generated from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$ for $p = d/m$, whose incidence matrix we denote by \mathbf{A} . If $s \geq 10d\ell/m$, then define $Q(r, \ell, s)$ as the event in which there exists an $r \times \ell$ sub-matrix of \mathbf{A} in which each row has at least s 1's. In this case,*

$$\mathbb{P}[Q(r, \ell, s)] \leq \binom{m}{r} \binom{n}{\ell} \exp\left(-\frac{rs \log((sm)/(d\ell))}{2}\right).$$

Using this lemma, we can ensure that *w.h.p.* ITERATED-COLOURING-ALGORITHM will not abort during phase one (Proposition 4.7) or two (Proposition 4.8) and thus conclude the proof of Theorem 1.3.

Proposition 4.7. *If ITERATED-COLOURING-ALGORITHM inputs a hypergraph drawn from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$ where $p = d/m$, then *w.h.p.* it does not abort during phase one, provided we assume that $\mu = dn/m \rightarrow \infty$ and $m \gg n$.*

Proof. Given $0 \leq i \leq t_1$, we say that round i is **good**, provided there are at most $n_i/17$ rows of H_i whose size is greater than $s_i := \beta\mu/16(i+2)^5$, where β satisfies (22). Otherwise, we say that the round is **bad**. Recall that $t_1 = \lg \mu$.

Now, if round i is good, then we claim that ITERATED-COLOURING-ALGORITHM does *not* abort in iteration i . To see this, it suffices to show that for n sufficiently large

$$\sum_{e \in E_i} \exp(-\lambda_e^2/16) \leq n_i/16,$$

where $n_i := |V_i| = n/2^i$, $\hat{f}_i := \hat{f}(i+2)^{-2}$ and $\lambda_e := \hat{f}_i/|e|^{1/2}$ for $e \in E_i$. Observe now that since the round is good, we get that

$$\begin{aligned} \sum_{e \in E_i} \exp(-\lambda_e^2/16) &= \sum_{\substack{e \in E_i: \\ |e| \leq s_i}} \exp(-\lambda_e^2/16) + \sum_{\substack{e \in E_i: \\ |e| > s_i}} \exp(-\lambda_e^2/16) \\ &\leq m \exp\left(-\frac{\hat{f}_i^2}{16s_i}\right) + n_i/17. \end{aligned}$$

On the other hand, since $\hat{f}_i := \sqrt{\mu \log(m/n)\beta}/(i+2)^2$,

$$\begin{aligned} m \exp\left(-\frac{\hat{f}_i^2}{16s_i}\right) &= m \exp(-\log(m/n)(i+2)) \\ &= m \left(\frac{n}{m}\right)^{i+2} \\ &= n \left(\frac{n}{m}\right)^{i+1} = o(n_i) \end{aligned}$$

where the last line follows since $(n/m)^{i+1} \ll 2^{-i}$, as $n \ll m$. Thus,

$$\sum_{e \in E_i} \exp(-\lambda_e^2/16) \leq (1 + o(1)) \frac{n_i}{17} \leq \frac{n_i}{16}.$$

We now must show that *w.h.p.* all of the rounds are good. Now, observe that if some round $1 \leq i \leq t_1$ is *bad*, then there exists an $(n_i/17) \times n_i$ sub-matrix of \mathbf{A} , say \mathbf{M} , in which each row of \mathbf{M} has greater than s_i 1's. In fact, since $n_{t_1} \leq n_i$ and $s_{t_1} \leq s_i$, we can take a sub-matrix of \mathbf{M} (and thus of \mathbf{A}) in which each row has at least s_{t_1} 1's, and whose size is $(n_{t_1}/17) \times n_{t_1}$. Thus, we observe the following claim:

- (1) If a bad round occurs, then there exists a $(n_{t_1}/17) \times n_{t_1}$ sub-matrix of \mathbf{A} in which each row has more than s_{t_1} 1's.

Let us define $Q(n_{t_1}/17, n_{t_1}, s_{t_1})$ as this latter event; namely, that there exists an $(n_{t_1}/17) \times n_{t_1}$ sub-matrix of \mathbf{A} in which each row has more than s_{t_1} 1's. In order to complete the proof, it suffices to show that *w.h.p.* $Q(n_{t_1}/17, n_{t_1}, s_{t_1})$ does not occur. Recall $n_{t_1} = n/\mu = m/d$ as the number of active vertices drops by exactly half in each round. Using Inequality (22), it follows that

$$s_{t_1} = \frac{\beta\mu}{16(t_1 + 2)^5} \geq \frac{\log \frac{m}{n} (\log \mu + 2)^5}{16(\lg \mu + 2)^5} \gg 10 = 10 \frac{dn}{\mu m} \geq 10 \frac{dn_{t_1}}{m}.$$

Thus, we can apply Lemma 4.6 to ensure the following (using $n_{t_1} = m/d$)

$$\begin{aligned} \mathbb{P}[Q(n_{t_1}/17, n_{t_1}, s_{t_1})] &\leq \binom{m}{n_{t_1}/17} \binom{n}{n_{t_1}} \exp\left(-\frac{n_{t_1}s_{t_1}}{34} \log\left(\frac{s_{t_1}m}{dn_{t_1}}\right)\right) \\ &\leq \binom{m}{n_{t_1}}^2 \exp\left(-\frac{n_{t_1}s_{t_1}}{34} \log s_{t_1}\right) \\ &\leq \left(\frac{me}{n_{t_1}}\right)^{2n_{t_1}} \exp\left(-\frac{n_{t_1}s_{t_1} \log s_{t_1}}{34}\right) \end{aligned}$$

where the inequalities follow since $m \gg n$, $\binom{m}{n_{t_1}} \leq (me/n_{t_1})^{n_{t_1}}$. Now,

$$\begin{aligned} \left(\frac{me}{n_{t_1}}\right)^{2n_{t_1}} &= \exp(2n_{t_1}(\log(m/n) + \log(ne/n_{t_1}))) \\ &= \exp(2n_{t_1}(\log(m/n) + \log(e\mu))), \end{aligned}$$

so

$$\mathbb{P}[Q(n_{t_1}/17, n_{t_1}, s_{t_1})] \leq \exp\left(-2n_{t_1} \left(\frac{s_{t_1} \log s_{t_1}}{68} - \log(m/n) - \log(e\mu)\right)\right).$$

Thus, by our assumption (22) on β , we get that

$$s_{t_1} = \frac{\beta\mu}{16(\lg \mu + 2)^5} \geq \frac{\log \frac{m}{n} (\log \mu + 2)^5}{16(\lg \mu + 2)^5} \geq \frac{1}{16} \log(m/n),$$

and

$$s_{t_1} = \frac{\beta\mu}{16(\lg \mu + 2)^5} \geq \frac{\mu}{16(\lg \mu + 2)^5},$$

as $\beta \geq 1$. The proposition follows as

$$\frac{\log(m/n) \log(\log(m/n)/16)}{16 \cdot 68} \gg \log(m/n),$$

and

$$\frac{\mu}{68 \cdot 16(\lg \mu + 2)^5} \gg \log(\mu e). \quad \square$$

Proposition 4.8. *If ITERATED-COLOURING-ALGORITHM inputs a hypergraph drawn from $\mathcal{H}(n, m, d)$ or $\mathbb{H}(n, m, p)$ where $p = d/m$, then *w.h.p.* it does not abort in phase two, provided $\mu = dn/m \rightarrow \infty$ as $n \rightarrow \infty$ and $m \gg n$.*

Proof. Suppose that $t_1 + 1 \leq i \leq t_2$ for $t_2 := \lg(10n/\hat{f}) + 1$. Recall $n_{t_2} = n/2(10n/\hat{f}) = \hat{f}/20$ and that during phase two, $\lambda_e = 0$ for each $e \in E_i$. Thus, we get that

$$\sum_{e \in E_i} \exp(-\lambda_e^2/16) = |E_i|.$$

On the other hand, in order for an edge of H to remain in E_i , it must have greater than \hat{f} vertices which lie in V_i thanks to step (1) of the ITERATED-COLOURING-ALGORITHM. As a result, if ITERATED-COLOURING-ALGORITHM aborts in round i , then H_i must have at least $n_i/16$ edges of size greater than \hat{f} . In particular, since $n_{t_2} \leq n_i$, this implies that the incidence matrix \mathbf{A} of H has an $(n_{t_2}/16) \times n_{t_2}$ sub-matrix in which each row has greater than \hat{f} 1's. Thus, if $Q(n_{t_2}/16, n_{t_2}, \hat{f})$ corresponds to the event in which \mathbf{A} has an $(n_{t_2}/16) \times n_{t_2}$ sub-matrix in which each row has greater than \hat{f} 1's, then we get the following claim:

- (1) If ITERATED-COLOURING-ALGORITHM aborts in some round $t_1 + 1 \leq i \leq t_2$, then $Q(n_{t_2}/16, n_{t_2}, \hat{f})$ must occur.

As a result, in order to show that *w.h.p.* ITERATED-COLOURING-ALGORITHM does *not* abort in any round it suffices to prove that $Q(n_{t_2}/16, n_{t_2}, \hat{f})$ does not occur *w.h.p.* Now, it follows that

$$\hat{f} \geq 10 \frac{d\hat{f}}{20m} = 10 \frac{dn_{t_2}}{m},$$

as $d \leq m$. Thus, we can apply Lemma 4.6 to ensure that

$$\begin{aligned} \mathbb{P}[Q(n_{t_2}/16, n_{t_2}, \hat{f})] &\leq \binom{m}{n_{t_2}/16} \binom{n}{n_{t_2}} \exp\left(-\frac{n_{t_2}\hat{f}}{32} \log\left(\frac{\hat{f}m}{dn_{t_2}}\right)\right) \\ &\leq \binom{m}{n_{t_2}/16} \binom{n}{n_{t_2}} \exp\left(-\frac{n_{t_2}\hat{f}}{32} \log\left(\frac{20m}{d}\right)\right), \end{aligned}$$

and so $\mathbb{P}[Q(n_{t_2}/16, n_{t_2}, \hat{f})]$ is upper bounded by

$$\exp\left(-2n_{t_2} \left(\frac{\hat{f}}{64} - \log(m/n) - \log(\mu e)\right)\right), \quad (23)$$

after applying the same simplifications as in Proposition 4.7. The proposition then follows by assumption (22) on β , as

$$\hat{f}/64 = \sqrt{\beta\mu \log(m/n)}/64 \geq \frac{\log(m/n) \log^5(\mu + 2)}{64} \gg \log m/n,$$

and

$$\frac{\log(m/n) \log^5(\mu + 2)}{64} \gg \log(\mu e). \quad \square$$

5. CONCLUSION AND OPEN PROBLEMS

We have lower bounded the discrepancy of the random hypergraph models $\mathbb{H}(n, m, p)$ and $\mathcal{H}(n, m, d)$ for the full parameter range in which $d \rightarrow \infty$ and $dn/m \rightarrow \infty$ where $p = d/m$. In the dense regime of $m \gg n$, we have provided asymptotically matching upper bounds, under the assumption that $d = pm \geq (m/n)^{1+\varepsilon}$ for some constant $\varepsilon > 0$. These upper bounds are algorithmic, and so the main question left open by our work is whether analogous upper bounds can be proven in the sparse regime of $n/\log n \ll m \ll n$. Our lower bounds suggest that the discrepancy is $\Theta(2^{-n/m} \sqrt{pn})$, and while we believe that a second moment argument could be used to prove the existence of such a colouring—particularly, in the edge-independent model $\mathbb{H}(n, m, p)$ —the partial colouring lemma does not seem to be of much use here. This leaves open whether such a colouring can be computed efficiently in this parameter range. If this is not possible, then ideally one could find a reduction

to a problem which is believed to be hard on average. One candidate may be the random-lattice problem of Ajtai [1] and Goldreich et al. [16], in which a random m by n matrix \mathbf{M} with *i.i.d.* entries from \mathbb{Z}_q is generated, and one wishes to compute a vector $\mathbf{x} \in \{0, 1\}^n$ such that $\mathbf{M}\mathbf{x} = 0$.

ACKNOWLEDGEMENTS

This work was initiated at the 2019 Graduate Research Workshop in Combinatorics, which was supported in part by NSF grant #1923238, NSA grant #H98230-18-1-0017, a generous award from the Combinatorics Foundation, and Simons Foundation Collaboration Grants #426971 (to M. Ferrara), #316262 (to S. Hartke) and #315347 (to J. Martin). We thank Aleksandar Nikolov for suggesting the problem, and Puck Rombach and Paul Horn for discussions and encouragements in the early stages of the project. Moreover, T. Masařík received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme Grant Agreement 714704. He completed a part of this work while he was a postdoc at Simon Fraser University in Canada, where he was supported through NSERC grants R611450 and R611368. X. Pérez-Giménez was supported in part by Simons Foundation Grant #587019.

REFERENCES

- [1] Miklós Ajtai. “Generating Hard Instances of Lattice Problems (Extended Abstract)”. In: *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing*. STOC ’96. Philadelphia, Pennsylvania, USA: Association for Computing Machinery, 1996, pp. 99–108. ISBN: 0897917855. DOI: 10.1145/237814.237838.
- [2] Dylan J. Altschuler and Jonathan Niles-Weed. “The discrepancy of random rectangular matrices”. In: *Random Structures & Algorithms* 60.4 (2022), pp. 551–593. DOI: <https://doi.org/10.1002/rsa.21054>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/rsa.21054>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/rsa.21054>.
- [3] Wojciech Banaszczyk. “Balancing vectors and Gaussian measures of n -dimensional convex bodies”. In: *Random Structures & Algorithms* 12.4 (1998), pp. 351–360. DOI: 10.1002/(SICI)1098-2418(199807)12:4<351::AID-RSA3>3.0.CO;2-S.
- [4] Nikhil Bansal, Daniel Dadush, and Shashwat Garg. “An Algorithm for Komlós Conjecture Matching Banaszczyk’s Bound”. In: *SIAM Journal on Computing* 48.2 (2019), pp. 534–553. DOI: 10.1137/17M1126795.
- [5] Nikhil Bansal, Aditi Laddha, and Santosh Vempala. “A Unified Approach to Discrepancy Minimization”. In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2022)*. Ed. by Amit Chakrabarti and Chaitanya Swamy. Vol. 245. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 1:1–1:22. ISBN: 978-3-95977-249-5. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2022.1. URL: <https://drops.dagstuhl.de/opus/volltexte/2022/17123>.
- [6] Nikhil Bansal and Raghu Meka. “On the discrepancy of random low degree set systems”. In: *Random Structures & Algorithms* 57.3 (June 2020), pp. 695–705. DOI: 10.1002/rsa.20935.
- [7] József Beck and Tibor Fiala. ““Integer-making” theorems”. In: *Discrete Applied Mathematics* 3.1 (Feb. 1981), pp. 1–8. DOI: 10.1016/0166-218x(81)90022-6.
- [8] Boris Bukh. “An Improvement of the Beck–Fiala Theorem”. In: *Combinatorics, Probability and Computing* 25.3 (2016), pp. 380–398. DOI: 10.1017/S0963548315000140.
- [9] Bernard Chazelle. *The Discrepancy Method: Randomness and Complexity*. Cambridge University Press, July 2000. DOI: 10.1017/cbo9780511626371.
- [10] William Chen, Anand Srivastav, and Giancarlo Travaglini, eds. *A Panorama of Discrepancy Theory*. Springer International Publishing, 2014. DOI: 10.1007/978-3-319-04696-9.

- [11] Persi Diaconis and Susan Holmes. *Stein’s Method: Expository Lectures and Applications*. Institute of Mathematical Statistics. Institute of Mathematical Statistics, 2004. ISBN: 978-0-940600-62-1. URL: <https://books.google.ca/books?id=3n-iQIU9LNEC>.
- [12] Esther Ezra and Shachar Lovett. “On the Beck-Fiala conjecture for random set systems”. In: *Random Structures & Algorithms* 54.4 (Nov. 2018), pp. 665–675. DOI: 10.1002/rsa.20810.
- [13] William Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons Inc., 1971.
- [14] Cole Franks and Michael Saks. “On the discrepancy of random matrices with many columns”. In: *Random Structures & Algorithms* 57.1 (2020), pp. 64–96. DOI: 10.1002/rsa.20909.
- [15] Alan Frieze and Michał Karoński. *Introduction to Random Graphs*. Cambridge University Press, 2015. DOI: 10.1017/CB09781316339831.
- [16] Oded Goldreich, Shafi Goldwasser, and Shai Halevi. “Collision-Free Hashing from Lattice Problems”. In: *Studies in Complexity and Cryptography. Miscellanea on the Interplay between Randomness and Computation: In Collaboration with Lidor Avigad, Mihir Bellare, Zvika Brakerski, Shafi Goldwasser, Shai Halevi, Tali Kaufman, Leonid Levin, Noam Nisan, Dana Ron, Madhu Sudan, Luca Trevisan, Salil Vadhan, Avi Wigderson, David Zuckerman*. Ed. by Oded Goldreich. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 30–39. ISBN: 978-3-642-22670-0. DOI: 10.1007/978-3-642-22670-0_5.
- [17] Rebecca Hoberg and Thomas Rothvoss. “A fourier-analytic approach for the discrepancy of random set systems”. In: *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM. 2019, pp. 2547–2556. DOI: 10.1137/1.9781611975482.156.
- [18] Avi Levy, Harishchandra Ramadas, and T. Rothvoss. “Deterministic Discrepancy Minimization via the Multiplicative Weight Update Method”. In: *Integer Programming and Combinatorial Optimization*. 2017, pp. 380–391. DOI: 10.1007/978-3-319-59250-3_31.
- [19] Shachar Lovett and Raghu Meka. “Constructive Discrepancy Minimization by Walking on the Edges”. In: *SIAM Journal on Computing* 44.5 (Jan. 2015), pp. 1573–1582. ISSN: 1095-7111. DOI: 10.1137/130929400.
- [20] Jiří Matoušek. *Geometric Discrepancy: An Illustrated Guide*. 1st ed. Algorithms and Combinatorics 18. Springer-Verlag Berlin Heidelberg, 1999. DOI: 10.1007/978-3-642-03942-3.
- [21] Lucas Pesenti and Adrian Vladu. “Discrepancy Minimization via Regularization”. In: *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 1734–1758. DOI: 10.1137/1.9781611977554.ch66. eprint: <https://epubs.siam.org/doi/pdf/10.1137/1.9781611977554.ch66>. URL: <https://epubs.siam.org/doi/abs/10.1137/1.9781611977554.ch66>.
- [22] Aditya Potukuchi. *Discrepancy in random hypergraph models*. 2018. arXiv: 1811.01491 [math.CO].
- [23] Aditya Potukuchi. “A Spectral Bound on Hypergraph Discrepancy”. In: *47th International Colloquium on Automata, Languages, and Programming (ICALP 2020)*. Ed. by Artur Czumaj, Anuj Dawar, and Emanuela Merelli. Vol. 168. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020, 93:1–93:14. ISBN: 978-3-95977-138-2. DOI: 10.4230/LIPIcs.ICALP.2020.93.
- [24] Irina Shevtsova. “An Improvement of Convergence Rate Estimates in the Lyapunov Theorem”. In: *Doklady Mathematics* 82 (Dec. 2010), pp. 862–864. DOI: 10.1134/S1064562410060062.
- [25] Paxton Turner, Raghu Meka, and Philippe Rigollet. “Balancing Gaussian vectors in high dimension”. In: *Conference on Learning Theory, COLT 2020, 9-12 July 2020, Virtual Event [Graz, Austria]*. Ed. by Jacob D. Abernethy and Shivani Agarwal. Vol. 125. Proceedings of Machine Learning Research. PMLR, 2020, pp. 3455–3486. URL: <http://proceedings.mlr.press/v125/turner20a.html>.
- [26] J. D. Williams. “An Approximation to the Probability Integral”. In: *Annals of Mathematical Statistics* 17.3 (Sept. 1946), pp. 363–365. DOI: 10.1214/aoms/1177730951.

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF TORONTO, TORONTO, ON, CANADA
Email address: `cmacrury@cs.toronto.edu`

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY,
PRAGUE, CZECH REPUBLIC & FACULTY OF MATHEMATICS, INFORMATICS AND MECHANICS, UNIVERSITY OF WAR-
SAW, WARSAW, POLAND & DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BC, CANADA
Email address: `masarik@kam.mff.cuni.cz`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN NE, USA
Email address: `lpai@huskers.unl.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN NE, USA
Email address: `xperez@unl.edu`