Forward-backward Contention Resolution Schemes for Fair Rationing

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Abstract

We use contention resolution schemes (CRS) to derive algorithms for the fair rationing of a single resource when agents have stochastic demands. We aim to provide ex-ante guarantees on the level of service provided to each agent, who may measure service in different ways (Type-I, II, or III), calling for CRS under different feasibility constraints (rank-1 matroid or knapsack). We are particularly interested in *two-order* CRS where the agents are equally likely to arrive in a known forward order or its reverse, which is motivated by online rationing at food banks. Indeed, for a mobile pantry driving along cities to ration food, it is equally efficient to drive that route in reverse on half of the days, and we show that doing so significantly improves the service guarantees that are possible, being more "fair" to the cities at the back of the route.

In particular, we derive a two-order CRS for rank-1 matroids with guarantee $1/(1+e^{-1/2}) \approx 0.622$, which we prove is tight. This improves upon the 1/2 guarantee that is best-possible under a single order [Ala14], while achieving separation with the $1 - 1/e \approx 0.632$ guarantee that is possible for random-order CRS [LS18]. Because CRS guarantees imply prophet inequalities, this also beats the two-order prophet inequality with ratio $(\sqrt{5}-1)/2 \approx 0.618$ from [ADK21], which was tight for single-threshold policies. Rank-1 matroids suffice to provide guarantees under Type-II or III service, but Type-I service requires knapsack. Accordingly, we derive a two-order CRS for knapsack with guarantee 1/3, improving upon the $1/(3 + e^{-2}) \approx 0.319$ guarantee that is best-possible under a single order [JMZ22]. To our knowledge, 1/3 provides the best-known guarantee for knapsack CRS even in the offline setting. Finally, we provide an upper bound of $1/(2 + e^{-1}) \approx 0.422$ for two-order knapsack CRS, strictly smaller than the upper bound of $(1 - e^{-2})/2 \approx 0.432$ for random-order knapsack CRS.

1 Introduction

Rationing a limited supply is a problem as old as society itself. In some circumstances, the demands to manifest are also uncertain, as agents are sojourners who come and go. Rationing with limited supply can be modelled as an online decision-making problem where the resource can either be put to good use serving present agents, or be rationed for future agents who may or may not show up.

Meanwhile, contention resolution schemes (CRS) are a modern tool for selecting a subset of agents, often online. They provide probabilistic guarantees to each agent for being selected, and operate under both a global budget constraint on the pool of agents selected, and local stochasticity in whether each agent can be feasibly selected. Since being introduced in the seminal works of

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[CVZ14, FSZ21], CRS have seen a burgeoning literature studying different feasibility structures, arrival patterns, and other variants, motivated by applications ranging from submodular optimization to Bayesian search to online stochastic matching.

In this paper we connect the two concepts, using CRS to derive rationing policies with guarantees on how well the demand of each agent will be served. In some sense, our approach is quite natural the global budget constraint in CRS captures the limited supply in rationing, while the local stochasticity in CRS captures the uncertain demand. That being said, problem-specific nuances arise from the different ways in demand service is measured in rationing, and our approach is able to use CRS to consider different rationing problems in a unified manner (see Subsection 1.1). The online rationing application also motivates a new "forward-backward" arrival pattern for CRS (see Subsection 1.2). Going full circle, our results for forward-backward CRS under the rank-1 matroid and knapsack feasibility structures (see Subsection 1.3) have implications beyond, improving twoorder prophet inequalities and random-order/offline knapsack CRS.

1.1 Rationing Preliminaries

Rationing problems have been studied under different service definitions and arrival patterns, with different goals in mind. We outline the differences and explain our approach to rationing.

Definition 1 (Setup). Agents $i \in [n] := \{1, \ldots, n\}$ have random demands $D_i \ge 0$ drawn independently from known distributions F_i with means $\mu_i > 0$. Each agent *i* receives a (random) allocation $Y_i \ge 0$, which must satisfy $\sum_{i=1}^{n} Y_i \le 1$ due to having a limited supply of 1. If agent *i* has demand *d* and receives allocation *y*, the *service* provided is given by $s_i(y, d)$, where s_i can take one of the three functional forms below. The expected service provided to an agent *i* is $\mathbb{E}[s_i(Y_i, D_i)]$.

- 1. The Type-I service function defines $s_i(y, d) = \mathbb{1}(y \ge d)$. Assuming $Y_i \le D_i$, we have $\mathbb{E}[s_i(Y_i, D_i)] = \Pr[Y_i = D_i]$.
- 2. The Type-II service function defines $s_i(y, d) = \min\{y, d\}/\mu_i$. Assuming $Y_i \leq D_i$, we have $\mathbb{E}[s_i(Y_i, D_i)] = \mathbb{E}[Y_i]/\mathbb{E}[D_i]$.
- 3. The Type-III service function defines $s_i(y, d) = \min\{y, d\}/d$. Assuming $Y_i \leq D_i$, we have $\mathbb{E}[s_i(Y_i, D_i)] = \mathbb{E}[Y_i/D_i]$.

For Type-III service, 0/0 is treated as 1, i.e. if demand is 0 then 100% service is trivially achieved.

We will always assume $Y_i \leq D_i$, which is without loss of generality if D_i is truthfully¹ revealed before Y_i has to be decided. Under this assumption, different settings arise depending on whether the D_i 's are revealed all at once or one by one.

Definition 2 (Offline vs. Online). In offline rationing, $(F_i)_{i \in [n]}$ is known in advance, $(D_i)_{i \in [n]}$ is revealed at the beginning, and then the algorithm decides $Y_i \in [0, D_i]$ for all *i* satisfying $\sum_{i=1}^n Y_i \leq 1$.

In online rationing, $(F_i)_{i \in [n]}$ is known in advance. The arrival permutation Λ , which could be randomly drawn from a known independent distribution, is revealed² at the beginning. Demands D_i are then revealed in order following Λ , after which $Y_i \in [0, D_i]$ must be immediately decided, with Y_i no greater than 1 minus the total allocation to agents who arrived before *i*.

¹There is an alternate literature that allows for demands to be misreported, which we mention in Subsection 1.5.

 $^{^{2}}$ This also fully reveals the identity of each arriving agent. See [EFGT23, EFT24] for some recent works that consider unknown orders or identities.

Past work has studied offline rationing, and online rationing when the arrival order Λ is fixed, with different goals in mind that we outline below.

- 1. Optimal Algorithms for Fairness: [LIS14] consider online rationing under Type-III service, where they maximize $\mathbb{E}[\min_{i \in [n]} Y_i/D_i]$, a Rawlsian fairness objective that aims to serve the worst-off agent as well as possible. They use dynamic programming to characterize the optimal algorithm for this objective, assuming independent demands.
- 2. Simple Algorithms with Fairness Guarantees: [MNR21] consider the same setting and objective as [LIS14], but allow for correlated demands, which makes the optimal algorithm intractable. Instead, they show that an elegant heuristic achieves the best-possible competitive ratio under worst-case correlated demands, relative to a benchmark measuring demand scarcity. They focus on Type-III service and prove that their algorithm is also best-possible for the objective min_{$i \in [n]$} $\mathbb{E}[Y_i/D_i]$, with the minimum outside the expectation.
- 3. Service Feasibility Determination: [JWZ23] consider offline rationing under all three types of service and potentially correlated demands. Instead of maximizing a fairness objective, they determine for given targets $(\beta_i)_{i \in [n]}$ whether it is possible to satisfy $\mathbb{E}[s_i(Y_i, D_i)] \geq \beta_i$ for all i, and if so, what is the allocation algorithm. They also investigate the minimum supply required to satisfy given targets $(\beta_i)_{i \in [n]}$, in combinatorial settings.

We now state our approach to rationing using CRS, and explain how it relates to goals (1)-(3) above. We establish the following general reduction.

Theorem 1.1. Under any combination of service functions from Definition 1, if $(\beta_i)_{i \in [n]}$ lies in a convex region (see Definition 4) which includes the vector $(\mathbb{E}[s_i(Y_i, D_i)])_{i \in [n]}$ for any (online or offline) algorithm, then an α -selectable CRS (defined in Subsection 1.2) for knapsack can be used to define an online rationing algorithm satisfying

$$\mathbb{E}[s_i(Y_i, D_i)] \ge \alpha \beta_i \qquad \forall i \in [n].$$
(1.1)

If every service function s_i is of Type-II or Type-III, then an α -selectable CRS for rank-1 matroids is sufficient to define an online rationing algorithm satisfying (1.1).

Our approach yields lower bounds on $\mathbb{E}[s_i(Y_i, D_i)]$ separately for each agent *i*, allowing us to study the objective $\min_i \mathbb{E}[s_i(Y_i, D_i)]$ for general types of service, or study feasibility determination.

- 1. For the goal of optimal algorithms, we note that with objective $\min_i \mathbb{E}[s_i(Y_i, D_i)]$, even under independent demands, dynamic programming is intractable³ and the approach of [LIS14] would not work. Our contribution here is to provide an α -approximation algorithm, where we can use our convex relaxation to compute an upper bound β on the objective achieved by any (online or offline) rationing algorithm, and then apply (1.1) with $\beta_1 = \cdots = \beta_n = \beta$ to establish an online algorithm with $\min_i \mathbb{E}[s_i(Y_i, D_i)] \ge \alpha\beta$, which is within an α -factor from optimality for some constant $\alpha \le 1$.
- 2. We are deriving simple algorithms with guarantees like in [MNR21], but able to provide approximations relative to the optimal algorithm, which their benchmark is not guaranteed to upper-bound (see [MNR21]). That being said, their benchmark is justified by their ability to handle correlated demands, and they are also able to handle both objectives $\mathbb{E}[\min_{i \in [n]} Y_i/D_i]$ and $\min_{i \in [n]} \mathbb{E}[Y_i/D_i]$.

³This is because with the minimum outside the expectation, the state is no longer captured by the minimum of $s_i(Y_i, D_i)$ over all agents *i* who have arrived so far. In fact, even the offline problem is highly non-trivial (see [JWZ23]).

3. Like [JWZ23], our approach applies to all three types of service and does not have a particular fairness objective in mind. We are essentially determining *online* service feasibility, where (1.1) shows that if service targets $(\beta_i)_{i \in [n]}$ are feasible in our relaxation, then targets $(\alpha \beta_i)_{i \in [n]}$ are feasible using an online algorithm. However, our approach does not provide a characterization of all feasible service vectors, the way that their approach does for the offline problem.

We now justify our objective of $\min_i \mathbb{E}[s_i(Y_i, D_i)]$ for goals (1) and (2) above, which is sometimes called "ex-ante" fairness. In contrast, [LIS14, MNR21] can both handle the "ex-post" fairness $\mathbb{E}[\min_i Y_i/D_i]$ under Type-III service. The latter is indeed well-justified by one-time allocation settings (e.g. ventilators during the COVID-19 pandemic), because the objective $\min_i Y_i/D_i$ can be empirically evaluated from one sample even if the true distributions are unknown. By contrast, our motivation comes more from repeated allocation settings, where it is acceptable for Y_i to be small one week if it is made up for in other weeks. In fact, in classical supply chain contexts where a distributor allocates Y_i^t to vendors *i* with demands D_i^t over weeks $t = 1, \ldots, T$, the fill rate is measured by what fraction of a vendor's overall demand is served (see e.g. [CM07, SLCB⁺05]), i.e.

Fill Rate of
$$i = \frac{Y_i^1 + \dots + Y_i^T}{D_i^1 + \dots + D_i^T} = \frac{\frac{1}{T}(Y_i^1 + \dots + Y_i^T)}{\frac{1}{T}(D_i^1 + \dots + D_i^T)}.$$
 (1.2)

In that sense, a reasonable⁴ fairness objective would be $\min_i \mathbb{E}[Y_i]/\mathbb{E}[D_i]$, i.e. ex-ante fairness under Type-II service, because the numerator in (1.2) is approximating $\mathbb{E}[Y_i]$ while the denominator is approximating $\mathbb{E}[D_i]$. Meanwhile, in contexts where a supplier is allocating to manufacturers, manufacturer *i* may only produce if their ordered stock D_i if met in entirety, in which case one should consider Type-I service $\Pr[Y_i = D_i]$ instead. Regardless, our approach works for ex-ante fairness maximization or service feasibility determination under arbitrary types of service.

Finally, we make a modeling contribution to online rationing, which is that permutation Λ can be random. It is well-known in online contention resolution schemes that averaging over a random order can improve guarantees; however, this observation was perhaps omitted in online rationing because for ex-post fairness objectives, a random order does not help. We are particularly interested in the random permutation where Λ is equally likely to be the order $1, \ldots, n$ or its reverse $n, \ldots, 1$, recently considered in [ADK21]. One motivation for this permutation is online rationing at food banks, where a mobile pantry⁵ drives along cities to ration food, and it does not increase⁶ driving distance to visit the cities in reverse order on half of the days. We now derive CRS's that are specialized for this forward-backward random permutation, even beating the guarantee for singleunit prophet inequality from [ADK21].

1.2 Contention Resolution Preliminaries

For $n \in \mathbb{N}$, we refer to [n] as a collection of *elements*. Let f be the *forward permutation*, i.e., f(i) = i for all $i \in [n]$, and b be the *backward permutation*, i.e., b(i) = n - (i - 1) for each $i \in [n]$. For $\sigma \in \{f, b\}$ and distinct $j, i \in [n]$, we denote $j <_{\sigma} i$, provided $\sigma(j) < \sigma(i)$.

⁴In fact, [MXX20] argue that ex-post fairness should go with Type-III service while ex-ante fairness should go with Type-II. They study only the combinations $\mathbb{E}[\min_i Y_i/D_i]$ and $\min_i \mathbb{E}[Y_i]/\mathbb{E}[D_i]$, which they call "short-run" and "long-run" fairness respectively.

⁵See [LIS14, SJBY23, BHS23] for more background on this application. All of these papers assume independent demands like we do, as daily shocks in food demand tend to be independent across locations.

⁶This would not be the case if the cities were visited in a uniformly random order, which is the more common model of random permutations in the CRS literature [AW18, LS18].

An input to the knapsack forward-backward contention resolution scheme (FB-CRS) problem is specified by $(n, (F_i)_{i=1}^n)$, where each F_i is distribution on $[0, 1] \cup \{\infty\}$. (Here ∞ is a special symbol, and we adopt the standard algebraic conventions involving it as an element of the extended reals.) We assume that each $i \in [n]$ independently draws a random size $S_i \sim F_i$. If $S_i < \infty$, then we refer to *i* as active. Otherwise, we refer to *i* as inactive. Note that using ∞ to distinguish between active/inactive elements is non-standard in the contention resolution literature; however it is convenient for our purposes, due to our fairness applications. In particular, in the reductions we present in Section 2, we shall think of $S_i = \infty$ as indicating that an agent *i* has demand too high to be worth servicing. It is also important for our fairness reduction to allow the active elements to have random sizes.

Let us assume that Λ is an independently drawn random permutation that is supported uniformly on $\{f, b\}$. A knapsack forward-backward contention resolution scheme (FB-CRS) is given $(n, (F_i)_{i=1}^n)$ as its input, and is also revealed the instantiation of Λ . It is then sequentially revealed the random sizes of the elements in the *increasing* order specified by Λ . That is, in *time* step $t \in [n]$, if $\Lambda(i) = t$ for $i \in [n]$, then it learns the instantiation of S_i . At this point, it makes an *irrevocable decision* on whether to accept *i*. Its output is a subset of accepted elements $I \subseteq [n]$ for which $\sum_{i \in I} S_i \leq 1$. Note that by definition, a knapsack FB-CRS can never accept an inactive element.

Given $\alpha \in [0, 1]$, we say that a knapsack FB-CRS is α -selectable on $(n, (F_i)_{i=1}^n)$, or that α is the selection guarantee of the FB-CRS on $(n, (F_i)_{i=1}^n)$, provided for all $i \in [n]$ and $s \in [0, 1]$,

$$\Pr[i \in I \mid S_i = s] \ge \alpha. \tag{1.3}$$

Here (1.3) is taken over the randomness in $(S_i)_{i=1}^n$, Λ , as well as any randomized decisions made by the FB-CRS. For knapsack constraints, we are interested in inputs with $\sum_{i=1}^n \mathbb{E}[S_i \cdot \mathbb{1}(S_i < \infty)] \leq 1$. If for a fixed $\alpha \in [0, 1]$, a knapsack FB-CRS satisfies (1.3) for *all* such inputs, then we refer to it as α -selectable.

An important special case is the forward-backward contention resolution problem for rank-1 matroids, which we hereafter refer to as single-unit CRS. An input for this problem is specified by (n, \boldsymbol{x}) , where $\boldsymbol{x} = (x_i)_{i=1}^n$ is a collection of probabilities (i.e., $0 \leq x_i \leq 1$ for all $i \in [n]$). In this case, $S_i \in \{1, \infty\}$ and $\Pr[S_i = 1] = x_i$ for each $i \in [n]$. Clearly at most one active element can be accepted by the FB-CRS, so we refer to it as a single-unit FB-CRS. The same definition (1.3) applies to a single-unit FB-CRS; however we are also interested in deriving results for inputs with $\sum_{i=1}^{n} \mathbb{E}[S_i \cdot \mathbb{1}(S_i < \infty)] = \sum_{i=1}^{n} x_i > 1$. For a fixed $\rho \geq 0$ and $\alpha \in [0, 1]$, we refer to a single-unit FB-CRS as (α, ρ) -selectable, provided it satisfies (1.3) for all inputs (n, \boldsymbol{x}) with $\sum_{i=1}^{n} x_i \leq \rho$.

1.3 Results for Forward-backward Contention Resolution

Theorem 1.2. There exists a single-unit FB-CRS which is $\left(\frac{\exp(\rho/2)}{1+\exp(\rho/2)\rho}, \rho\right)$ -selectable for all $\rho \ge 0$. In particular, if $\rho = 1$, then the selection guarantee is at least $1/(1 + e^{-1/2}) > 0.622$.

Theorem 1.3. Fix $\rho \ge 0$. Then, no single-unit FB-CRS is more than $\left(\frac{\exp(\rho/2)}{1+\exp(\rho/2)\rho},\rho\right)$ -selectable.

For single-unit adversarial and random-order CRS's, the tight selection guarantees are respectively 1/2, due to [Ala14], and 1 - 1/e, due to [LS18]. Our tight guarantee of $1/(1 + e^{-1/2})$ for FB-CRS is sandwiched strictly in-between these values.

CRS guarantees apply directly to the prophet inequality setting, in which agents have valuations drawn from known independent distributions, and the objective is to maximize the expected valuation of the accepted agent. By a standard reduction [FSZ21], our FB-CRS implies an online algorithm that accepts an agent with valuation at least $1/(1 + e^{-1/2}) > 0.622$ times the maximum valuation in expectation, improving upon the two-order prophet inequality guarantee of $(\sqrt{5}-1)/2 \approx 0.618$ from [ADK21]. Interestingly, their guarantee is achieved using a static threshold, and tight⁷ for this class of policies. This shows that in this two-order setting, the tight prophet inequality for static thresholds $((\sqrt{5}-1)/2)$ differs from the tight CRS guarantee $(1/(1 + e^{-1/2}))$, whereas in the fixed-order and random-order settings, the two tight guarantees coincide at 1/2 and 1 - 1/e respectively [SC84, Ala14, EHKS18].

Theorem 1.4. There exists a knapsack FB-CRS which is 1/3-selectable.

As previously mentioned, 1/3 improves on the $1/(3+e^{-2})$ selection guarantee which was proven to be tight for adversarial arrivals by [JMZ22]. While the selection guarantee of Theorem 1.4 is derived in the forward-backward arrival model, it is actually the best known bound even for offline⁸ CRS. It appears difficult to leverage offline selection or random-order arrivals under knapsack constraints, but we manage to leverage the forward-backward random order to improve upon the result of [JMZ22] (see Subsection 1.4 for more details), which is why our result appears to be state-of-the-art even for these "easier" arrival models.

Theorem 1.5. No knapsack FB-CRS is more than $\frac{1}{2+e^{-1}}$ -selectable.

We prove Theorem 1.5 on the same knapsack input considered by [JMZ22] to derive a $(1-e^{-2})/2$ hardness result for offline CRS's. While we use the same input, we reduce the analysis to a related single-unit input. Our bound then follows by taking $\rho = 2$ in Theorem 1.3.

1.4 Technical Overview

Reduction. We discuss allocating to Type-III service functions using a single-unit CRS, which is the most challenging case in proving Theorem 1.1 and is also the service function used in [LIS14, MNR21]. Given a service target β_i , our convex region resembles an "ex-ante" relaxation that computes, separately for agent *i*, the most efficient way to achieve target β_i . This entails serving agent *i* whenever their demand D_i does not exceed some threshold d_i (possibly with randomized tiebreaking), and if so, serving them as much as possible, i.e. serving them min{ $D_i, 1$ }. From this one can compute an ideal, minimal amount x_i to allocate to *i* in expectation, which we input to the single-unit CRS as their active probability. More precisely, d_i and x_i satisfy

$$\mathbb{E}\left[\frac{\min\{D_i,1\}}{D_i}\mathbb{1}(D_i \le d_i)\right] = \beta_i; \qquad \mathbb{E}[\min\{D_i,1\}\mathbb{1}(D_i \le d_i)] = x_i.$$
(1.4)

Our rationing algorithm then allocates to every agent *i* exactly $c_{\sigma}(i)x_i$ supply in expectation, where $c_{\sigma}(i)$ is the probability they are selected by the CRS conditional on arrival order σ . It uses a simple threshold rule: agent *i* is allocated min $\{D_i, R_i, \tau_i\}$ whenever $D_i \leq d_i$, where $R_i \in [0, 1]$ is the remaining supply, and $\tau_i \in [0, 1]$ is tuned so that

$$\mathbb{E}[\min\{D_i, R_i, \tau_i\} \mathbb{1}(D_i \le d_i)] = c_\sigma(i) x_i.$$

$$(1.5)$$

We show that this can be inductively maintained as R_i dwindles, by using concavity to argue that the worst-case distribution of R_i is bimodal, supported on $\{0,1\}$. In this case, feasibility of selection

⁷The hardness result of [ADK21] in fact is stronger, and applies to static threshold free-order prophet inequalities whose random processing order has support size at most $O(\log n)$, where n is the number of elements.

⁸Offline CRS's learn the instantiations of $(S_i)_{i=1}^n$ all at once, and were the original model of contention resolution introduced by [CVZ14].

probabilities $(c_{\sigma}(i))_{i \in [n]}$ in the CRS (under order σ) coincides exactly with there existing $(\tau_i)_{i \in [n]}$ that preserve the expected allocations in (1.5).

Finally, we need to show that (1.5) implies

$$\mathbb{E}\left[\frac{\min\{D_i, R_i, \tau_i\}}{D_i}\mathbb{1}(D_i \le d_i)\right] \ge c_{\sigma}(i)\beta_i,\tag{1.6}$$

where the relationship between β_i , d_i , and x_i is implicit, as defined in (1.4). If we assume that $D_i \leq 1$, then a single application of the FKG inequality establishes (1.6). The general case requires a non-trivial in-between step showing that truncating demands to 1 can only correlate terms in our favor, essentially still following the FKG inequality.

Single-unit FB-CRS For the purposes of this technical overview, let us assume that (n, x) is a single-unit FB-CRS input with $\sum_{i=1}^{n} x_i = 1$ (i.e., we explain the proof with $\rho = 1$). In order to prove Theorems 1.2 and 1.3, our approach is to first characterize the instance-optimal selection guarantee an FB-CRS can attain on (n, x). This can be done through the following linear program (LP):

$$\begin{array}{ll} \text{maximize} & \min_{1 \leq i \leq n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2 \\ \text{subject to} & c_{\sigma}(i) \leq 1 - \sum_{j < \sigma^{i}} x_{j} \cdot c_{\sigma}(j) & \forall i \in [n], \sigma \in \{\mathsf{f}, \mathsf{b}\} \\ & c_{\sigma}(i) \geq 0 & \forall i \in [n], \sigma \in \{\mathsf{f}, \mathsf{b}\} \end{array}$$

in which variable $c_{\sigma}(i)$ represents the probability of accepting *i* conditional on it being active, under permutation σ (see Subsection 3.1 for further details).

However, the challenge is to identify the minimum of LPOPT(n, x) over all inputs (n, x) with $\sum_{i=1}^{n} x_i = 1$. Following the precedent in previous papers such as [Ala14, LS18, JMZ22], it is reasonable to "guess" that the worst-case instance occurs when $x_i = 1/n$ for each $i \in [n]$ and $n \to \infty$. (While we show this implicitly in Section 3, in Appendix B.5 we provide a direct proof of this.) Our proof strategy is to look at the structure of the optimal LP solution on this "worst-case" instance, to *inform a general method of constructing a feasible solution* (that is not necessarily optimal) on any instance, whose objective can be analytically lower-bounded by $1/(1 + e^{-1/2})$.

When $x_i = 1/n$ for each $i \in [n]$, there are two key simplifying assumptions that we can make involving an optimal LP solution. The first uses the uniformity of the x_i values, and the second allows us to remove the minimum in the objective:

- 1. $c_{f}(i) = c_{b}(n (i 1))$ for each $i \in [n]$;
- 2. $c_{f}(i) + c_{b}(i) = c_{f}(1) + c_{b}(1)$ for each $i \in [n]$.

By restricting to such solutions, and taking $n \to \infty$, we can reformulate our LP in terms of the following optimization problem involving a continuous function $\phi : [0, 1] \to [0, 1]$:

maximize
$$(\phi(0) + \phi(1))/2$$
 (cont-OPT)
subject to $\phi(z) + \phi(1-z) = \phi(0) + \phi(1)$ $\forall z \in [0,1]$ (1.7)
 $\phi(z) \le 1 - \int^z \phi(\tau) d\tau$ $\forall z \in [0,1]$ (1.8)

$$\begin{array}{c} J_0\\ \phi(z) \geq 0 \end{array} \qquad \forall z \in [0,1], \sigma \in \{\mathsf{f},\mathsf{b}\} \end{array}$$

(Here we have applied $c_f(i) = \phi(i/n)$ and $c_b(i) = \phi(1 - (i-1)/n)$.)

Our goal is to solve cont-OPT. Constraint (1.8) implies that any solution ϕ should be nonincreasing on [0,1]. To that end, one natural class of functions to try to optimize over is linear functions. Doing so yields $\phi(z) = 2/\sqrt{5} - (1 - 1/\sqrt{5})z$, in which case ϕ is a feasible solution to (cont-OPT) that satisfies $(\phi(0) + \phi(1))/2 = \frac{1}{2}(\sqrt{5} - 1) \approx 0.618$. Coincidentally, this matches the golden ratio bound for two-order prophet inequalities from [ADK21], that was obtained through a threshold analysis unrelated to CRS.

To go beyond their golden ratio bound, we guessed that a better solution to cont-OPT should make (1.8) hold as equality for all $z \in [1/2, 1]$. Combined with (1.7), this allowed us to identify a collection of piece-wise defined functions which are feasible. By optimizing over all such functions, this led to following solution, which satisfies $(\phi(0) + \phi(1))/2 = 1/(1 + e^{-1/2}) \approx 0.622$:

$$\phi(z) := \begin{cases} \frac{2e^{1/2} - e^z}{1 + e^{1/2}} & \text{if } z \le 1/2, \\ \frac{e^{1-z}}{1 + e^{1/2}} & \text{if } 1/2 < z \le 1. \end{cases}$$
(1.9)

In Subsection 3.1, we formally show how to use (1.9) to complete the proof of Theorem 1.2.

To prove the negative result Theorem 1.3, for the family of inputs described by $x_i = 1/n$ for all *i*, we show that LPOPT(*n*, *x*) is upper-bounded by $(1 + o(1))/(1 + e^{-1/2})$, where o(1) tends to 0 as $n \to \infty$. We upper-bound LPOPT(*n*, *x*) using weak duality and the challenge lies in identifying a dual feasible solution for this family of inputs whose objective value can be analyzed to be $(1 + o(1))/(1 + e^{-1/2})$ as $n \to \infty$. Our strategy is similar to the description above of how we lower-bounded LPOPT, which is to study the continuous analogue as $n \to \infty$. Interestingly, for the dual we must modify its solution, due to a discrepancy in the optimal solution for the continuous problem vs. any finite *n*. In particular, for finite *n*, the solution has a "discontinuity" where it must take an enormous value at n/2 to ensure feasibility. The details can be found in Subsection 3.2.

Knapsack FB-CRS For the purpose of this overview, let us assume that our knapsack input $(n, (F_i))_{i=1}^n$ has deterministic sizes. That is, there exists probabilities $(x_i)_{i=1}^n$ and (non-negative) sizes $(s_i)_{i=1}^n$, such that $S_i \in \{s_i, \infty\}$, and $\Pr[S_i = s_i] = x_i$ for each $i \in [n]$. More, assume that $\sum_{i=1}^n s_i x_i = 1$.

When designing a knapsack FB-CRS, one natural approach is to first split the elements into low and high elements, based on their associated size s_i . Concretely, let $L = \{i \in [n] : s_i \leq 1/2\}$ and $H = \{i \in [n] : s_i > 1/2\}$. Moreover, assume that $\sum_{i \in L} s_i x_i = \sum_{i \in H} s_i x_i = 1/2$. Clearly an FB-CRS can select at most one element from H, and so since $\frac{\sum_{i \in H} x_i}{2} \leq \sum_{i \in H} s_i x_i = 1/2$, we can immediately ensure a selection guarantee of $1/(1 + e^{-1/2}) \approx 0.622$ on H by applying Theorem 1.2 with $\rho = 1$. On the other hand, since the elements of L have deterministic sizes at most 1/2, it is possible to extend our approach to proving Theorem 1.2 to get a selection guarantee of $1/(1 + e^{-1/2})$ on L. Unfortunately, we have to balance prioritizing L or H, and so the best selection guarantee attainable in this way is $1/(2(1 + e^{-1/2})) \approx 0.311$. This is strictly worse than $1/(3 + e^{-2}) \approx 0.319$, the selection guarantee attained by [JMZ22] for a single arrival order.

To beat $1/(3 + e^{-2})$, we instead use the invariant-based argument in [JMZ22] to characterize feasible probabilities $(c_{\rm f}(i), c_{\rm b}(i))_{i=1}^n$ (see Definition 5), for which it is possible for an online algorithm to accept *i* with probability $c_{\sigma}(i)$ conditional on *i* being active and $\Lambda = \sigma$, for all $i \in [n]$ and $\sigma \in \{f, b\}$. This would lead to a selection guarantee of $\min_{i \in [n]} (c_{\rm f}(i) + c_{\rm b}(i))/2$, after averaging over Λ . We repeat the proof strategy throughout the paper of using a continuous function ϕ to define these feasible selection probabilities. Fortunately, in this case optimizing over linear functions ϕ suffices to get a clean bound of 1/3, that beats the single-order guarantee of $1/(3 + e^{-2})$ from [JMZ22]. Interestingly, our actual algorithm follows an arguably simpler heuristic than [JMZ22], which is also sufficient to recover their $1/(3 + e^{-2})$ result. After conditioning on $\Lambda = \sigma \in \{f, b\}$, the main observation is that when deciding whether to accept *i*, we should prioritize the acceptance of *i* on feasible sample paths where at least one other (non-zero) element was previously accepted. In other words, we avoid accepting *i* on sample paths where nothing else has been previously accepted, only doing so if necessary. Consequently, our algorithm ends up being stated differently than the algorithm from [JMZ22] that sets a threshold on size, leading to an arguably simpler proof, and allowing us to establish the guarantee of 1/3 in our forward-backward setting.

See Subsection 4.1 for details of this knapsack FB-CRS. Our negative result for knapsack FB-CRS, in Subsection 4.2, recycles the result of Theorem 1.3 for a general ρ .

1.5 Further Related Work

Online rationing. We refer to [SJBY23] for a recent work describing the mobile food pantry application, which also discusses competing fairness objectives and how they can be simultaneously captured by Nash social welfare. In food rationing applications, it is also natural if the initial supply can perish over time, as studied in [BHS23]. These papers contain more extensive references to the online fair resource allocation literature.

In general, we have already mentioned the technical results most related to ours in Subsection 1.1, but should further mention the concurrent work [SSX24]. They derive a competitive ratio of 1/2 for online rationing under ex-ante Type-III service, given a fixed arrival order. Our work focuses on forward-backward arrival order and shows how to beat 1/2 under this assumption, while also applying to other types of service. By contrast, their work also applies to ex-post fairness objectives, which we cannot handle.

Contention resolution. Beginning with the works of [CVZ14, GN13, FSZ21], CRS's have found broad applications as a general purpose tool in online and stochastic optimization. We refer to [Dug20, Dug22] for a connection to the matroid secretary problem, [PW24, NSW25] for applications to stationary prophet inequalities and prophet approximation, and [FLT⁺24] for an application to matroid prophet inequalities with limited sample access.

CRS's have been studied for a wide range of constraint systems. In this paper we focus on single-unit and knapsack, but there is also a lot of work for k-unit selection [Ala14, JMZ22, DW24], matroids [FSZ21, LS18, Dug20, FLT⁺24], and matchings [EFGT22, MMG24, PRSW22, FTW⁺21]. How forward-backward CRS fits relative to adversarial-order and random-order CRS could be investigated for all of these constraint systems.

CRS's have been recently adapted to handle correlation in the elements' activeness [QS22, GHKL24, DKP24, MMZ24, BMMP24], expanding their applicability as a general purpose tool.

Prophet inequality variants. There is a vast literature on prophet inequalities beyond adversarial arrivals. Classical variants include random-order [EHLM17, ACK18, CSZ19] in which the state-of-the-art is 0.688 due to [CHLT25]. An important special case is when valuations are drawn from identical distributions, in which case the tight guarantee is ≈ 0.745 [CFH⁺21, Ker86]. In the variant where the arrival order is chosen by the algorithm, there has been substantial recent progress (see [PT22, BC23, GMTS24]).

We study a random arrival order beyond these basic models, inspired by [ADK21]. We should note that there is also a line of work [KKN15, HKKO22] studying random arrival orders beyond the basic ones in settings with unknown distributions. Fairness in optimal stopping in the prophet setting has also been recently considered in [AK22], [CCDNF21]. These papers respectively study individual and group-level fairness constraints that are unrelated to our fairness notions based on CRS.

Incentive-compatible rationing and supply chain management. Classical economics literature has considered the offline rationing of a single infinitely-divisible resource when agents have unknown single-peaked preferences. A celebrated prior-free incentive-compatible mechanism was developed in [Spr91], which has been generalized in [BJN97]. Meanwhile, it is well-known in classical supply chain literature that retailers may exaggerate demands under typical proportional allocation rules used by distributors in times of shortage [LPW97, CL99]. Without disincentive for overallocation, this is guaranteed to lead to cheap talk that can manifest in practice [BYDH19], although there are solutions under repeated games [BGS19].

Our notions of Type-I, II, and III service are well-established in supply chain literature (see [JWZ23] and the references therein). We should note that the original motivation in [JWZ23] and several related papers [ZZCT18, LCC⁺19] is inventory pooling, i.e. the increased feasibility of service targets if supply is divided after seeing demand realizations instead of having dedicated stockpiles beforehand.

2 Reducing from Rationing to CRS

Our goal in this section is to prove Theorem 1.1. First we define the notion of quantiles.

Definition 3. For a distribution F over non-negative reals, define its inverse CDF over $q \in [0, 1]$ to be $F^{-1}(q) := \inf\{d : q \leq F(d)\}$. Define each agent $i \in [n]$ to draw an independent quantile Q_i uniformly from [0,1], and then have demand $D_i = F_i^{-1}(Q_i)$.

Definition 3 provides an equivalent method of generating demands. If demands cannot be generated like this and one only observes D_i (drawn from a known F_i) instead, then quantile Q_i can be assigned as follows: if F_i has no discrete mass on the realized value of D_i , then $Q_i := F_i(D_i)$; otherwise, if F_i has mass δ on D_i , then assign Q_i uniformly at random from $(F_i(D_i) - \delta, F_i(D_i)]$. The resulting distribution of quantiles is uniform modulo a set of measure 0.

Using the inverse CDF, we now define the convex region referenced in Theorem 1.1.

Definition 4. Given service types from Definition 1, define the *ex-ante feasible region* to be the collection of service targets $(\beta_i)_{i \in [n]} \in [0, 1]^n$ for which there exist $(q_i)_{i \in [n]} \in [0, 1]^n$ satisfying

$$\sum_{i=1}^{n} \int_{0}^{q_{i}} \min\{F_{i}^{-1}(q), 1\} dq \le 1$$
(2.1)

$$\int_{0}^{q_{i}} s_{i} \left(\min\{F_{i}^{-1}(q), 1\}, F_{i}^{-1}(q) \right) dq = \beta_{i} \qquad \forall i \in [n].$$
(2.2)

Remark 1. Constraints (2.1)–(2.2) can be simplified under the specific service types from Definition 1, which also allows us to see that the ex-ante feasible region is *convex*. We first note that the left-hand side (LHS) of constraints (2.1) are convex in q_i , because the derivative is min $\{F_i^{-1}(q_i), 1\}$ which is non-decreasing in q_i . Therefore, (2.1) induces a convex feasible region for $(q_i)_{i \in [n]}$. We now show that (2.2) also induces a convex feasible region for each q_i , which when taken in intersection with the convex region from (2.1) would establish that the ex-ante feasible region is convex. We separately consider each service type: 1. If s_i is a Type-I service function, i.e. $s_i(y,d) = \mathbb{1}(y \ge d)$, then (2.2) becomes

$$\beta_i = \int_0^{q_i} \mathbb{1}(F_i^{-1}(q) \le 1) dq = \min\{q_i, F_i(1)\}$$
(2.3)

which is equivalent to the convex constraints $\beta_i \leq q_i$ and $\beta_i \leq F_i(1)$;

2. If s_i is a Type-II service function, i.e. $s_i(y,d) = \min\{y,d\}/\mu_i$, then (2.2) becomes

$$\beta_i = \frac{1}{\mu_i} \int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq \tag{2.4}$$

which induces a convex region for q_i because it consists of a single point (note that this also leads to the term $\int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq$ in (2.1) simplifying to $\beta_i \mu_i$);

3. If s_i is a Type-III service function, i.e. $s_i(y, d) = \min\{y, d\}/d$, then the LHS of (2.2) becomes

$$\int_{0}^{q_{i}} \min\{1, \frac{1}{F_{i}^{-1}(q)}\} dq$$
(2.5)

which is concave in q_i , inducing a convex region.

We also note that for discrete distributions, (2.1)-(2.2) simplify to linear constraints and the feasible region is a polytope.

Intuitively, (2.2) is saying that if agent *i* is granted the entire supply of 1 whenever their quantile lies below q_i , i.e. allocated min $\{D_i, 1\}$ when $Q_i \leq q_i$, then the expected service provided would be

$$\mathbb{E}[s_i(\min\{D_i, 1\}, D_i)\mathbb{1}(Q_i \le q_i)] = \int_0^{q_i} s_i\left(\min\{F_i^{-1}(q), 1\}, F_i^{-1}(q)\right) dq = \beta_i.$$
(2.6)

Meanwhile, (2.1) is saying that the total expected supply allocated this way cannot exceed 1.

Given these interpretations, the following lemma is straight-forward to prove. We do need the property of all service functions from Definition 1 that the service provided relative to supply allocated is weakly decreasing in the demand, i.e. it is most "supply-efficient" to allocate to low demands, or equivalently low quantiles. We note this is opposite in mechanism design and prophet inequalities, where one would like to allocate to high valuations (see [Har13, Ch. 3]).

Lemma 2.1 (proof in Appendix A.1). Under the service types from Definition 1, for any (online or offline) algorithm, the vector $(\mathbb{E}[s_i(Y_i, D_i)])_{i \in [n]}$ lies in the ex-ante feasible region.

Given $(\beta_i)_{i \in [n]}$ in the ex-ante feasible region, we can use a CRS to define an online rationing algorithm. The reduction is quite obvious for the statement about knapsack CRS in Theorem 1.1. Indeed, define element *i* to be active with size $S_i = \min\{D_i, 1\}$ if $Q_i \leq q_i$, and inactive otherwise. Individual sizes are at most 1 by definition, and the expected total size of active elements is at most 1 by (2.1). Therefore, we can query a knapsack CRS and set $Y_i = S_i$ (using up capacity S_i) whenever the CRS says to accept element *i*, and set $Y_i = 0$ otherwise. By the CRS guarantee, we have $Y_i = S_i$ with probability at least α conditional on $Q_i = q$, for any $q \leq q_i$. Therefore, $\mathbb{E}[s_i(Y_i, D_i)] \geq \alpha \cdot \mathbb{E}[s_i(S_i, D_i)\mathbb{1}(Q_i \leq q_i)] = \alpha \beta_i$ by (2.6), establishing (1.1) in Theorem 1.1.

The reduction to single-unit CRS, however, is non-trivial and requires defining fractional allocations based on a CRS that accepts or rejects. (Although the knapsack reduction works for all service types, selection guarantees are much better for single-unit CRS, and hence we should use the latter if all service functions are of Type-II or III.) To use the single-unit CRS, given $(\beta_i)_{i \in [n]}$ in the ex-ante feasible region, we take corresponding values of $(q_i)_{i \in [n]}$, and define

$$x_i := \int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq \qquad \forall i \in [n]$$
(2.7)

as the activeness probabilities in the CRS, which satisfy $\sum_{i=1}^{n} x_i \leq 1$ by (2.1). We obtain conditional acceptance probabilities $c_{\sigma}(i)$ for each element *i* under each permutation σ (for FB-CRS, this would be returned by the LP-SI described in Section 3). The way in which these conditional probabilities are used to derive fractional allocations back in the rationing problem is described in Algorithm 1.

Algorithm 1	Using a Single-Unit CRS	to define a Ra	ationing Algorith	m			
\mathbf{T} $()$	(1, 1, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3,	1 (())	= [0, 1]n	1 1	11	ODO C	

Input: $(q_i)_{i \in [n]}$ satisfying (2.1)–(2.2), and $(c_{\sigma}(i))_{i \in [n]} \in [0, 1]^n$ returned by the CRS for every arrival order σ in the support of Λ (the input to the CRS is determined via (2.7))

Output: (random) online allocations Y_1, \ldots, Y_n satisfying $\mathbb{E}[s_i(Y_i, D_i)] \ge \mathbb{E}[c_{\Lambda}(i)]\beta_i$ for all i1: Initialize Rem = 1 \triangleright Remaining supply

- 2: Observe realized permutation Λ and call it σ
- 3: for *i* arriving in increasing order of σ do
- 4: Observe quantile Q_i and demand $D_i = F_i^{-1}(Q_i)$ \triangleright see Definition 3
- 5: **if** $Q_i \leq q_i$ **then** 6: Set $Y_i = \min\{D_i, \text{Rem}, \tau_i\}$, where τ_i is calibrated so that \triangleright Only allocate to i if $Q_i \leq q_i$ \triangleright We will prove τ_i exists

$$\mathbb{E}_{\text{Rem}}\left[\int_{0}^{q_{i}}\min\{F_{i}^{-1}(q), \text{Rem}, \tau_{i}\}dq \middle| \Lambda = \sigma\right] = c_{\sigma}(i)x_{i}$$
(2.8)

7: Update
$$\operatorname{Rem} = \operatorname{Rem} - Y_i$$

Intuitively, Algorithm 1 only considers allocating to an agent *i* if $Q_i \leq q_i$, because as mentioned earlier, it is most supply-efficient to allocate to low demands. The amount allocated is limited by both D_i and Rem, but we would like to further limit it to a threshold τ_i , to preserve the expected allocation of the CRS which is $c_{\sigma}(i)x_i$ (conditional on $\Lambda = \sigma$). We emphasize that the expectation in (2.8) is taken over the randomness in Rem; i.e., the algorithm ignores the present value of Rem and considers the distribution of remaining supply over all sample paths to do the calibration.

Lemma 2.2. Suppose every service function s_i is of Type-II or Type-III. Then given $(\beta_i)_{i \in [n]}$ in the convex ex-ante feasible region and an α -selectable single-unit CRS, Algorithm 1 provides expected service at least $\alpha\beta_i$ to every agent *i*.

Proof of Lemma 2.2. We fix σ throughout the proof and all statements are made conditioning on $\Lambda = \sigma$. Let R_i denote the remaining supply Rem when agent *i* arrives (under this σ). For *i* in increasing order of σ , we inductively establish that τ_i exists. We then show that the resulting allocation satisfies $\mathbb{E}[s_i(Y_i, D_i) \mid \Lambda = \sigma] \geq c_{\sigma}(i)\beta_i$. Since an α -selectable FB-CRS implies that $\mathbb{E}[c_{\Lambda}(i)] \geq \alpha$ by definition, this would provide expected service $\mathbb{E}[s_i(Y_i, D_i)] \geq \alpha\beta_i$ to every agent *i*, as desired.

In the base case, where *i* is first to arrive and hence Rem = 1, the LHS of (2.8) continuously decreases from x_i to 0 as τ_i decreases from 1 to 0. Because $c_{\sigma}(i) \in [0, 1]$, the mean value theorem ensures the existence of a value $\tau_i \in [0, 1]$ at which equality in (2.8) is achieved. Equality in (2.8) can be equivalently written as

$$\mathbb{E}[Y_i \mid \Lambda = \sigma] = \mathbb{E}[\min\{D_i, R_i, \tau_i\}\mathbb{1}(Q_i \le q_i) \mid \Lambda = \sigma] = c_\sigma(i)x_i.$$
(2.9)

We now show that (2.9) can be maintained. By induction, $\mathbb{E}[Y_j \mid \Lambda = \sigma] = c_{\sigma}(j)x_j$ for all j who came before i under permutation σ , and hence $\mathbb{E}[R_i \mid \Lambda = \sigma] = 1 - \sum_{j < \sigma^i} c_{\sigma}(j)x_j$. Now, note that $\int_0^{q_i} \min\{F_i^{-1}(q), r\}dq$ is a concave function of r. Therefore, to minimize $\mathbb{E}[\int_0^{q_i} \min\{F_i^{-1}(q), R_i\}dq \mid \Lambda = \sigma]$ over all distributions of $R_i \in [0, 1]$ with a fixed mean, R_i should be bimodally distributed, i.e. $R_i = 1$ with probability $1 - \sum_{j < \sigma^i} c_{\sigma}(j)x_j$ and $R_i = 0$ otherwise. This implies

$$\mathbb{E}\left[\int_{0}^{q_{i}} \min\{F_{i}^{-1}(q), R_{i}\}dq \middle| \Lambda = \sigma\right] \ge \left(1 - \sum_{j < \sigma i} c_{\sigma}(j)x_{j}\right) \int_{0}^{q_{i}} \min\{F_{i}^{-1}(q), 1\}dq$$
$$\ge c_{\sigma}(i) \int_{0}^{q_{i}} \min\{F_{i}^{-1}(q), 1\}dq ,$$

where the final inequality holds because $1 - \sum_{j < \sigma^i} c_{\sigma}(j) x_j \leq c_{\sigma}(i)$ is a necessary constraint for the conditional acceptance probabilities of a single-unit CRS under any arrival order σ (see constraint (3.1)). Because $\int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq = x_i$ by definition (2.7), this proves that the LHS of (2.8) is at least $c_{\sigma}(i)x_i$ when $\tau_i = 1$, so we can again apply the mean value theorem to justify the existence of a value $\tau_i \in [0, 1]$ at which equality in (2.8) is achieved. This completes the induction and establishes (2.9) for every agent *i*.

Finally, we show that (2.9) implies $\mathbb{E}[s_i(Y_i, D_i) \mid \Lambda = \sigma] \ge c_{\sigma}(i)\beta_i$ as long as s_i is the Type-II or Type-III service function, which would complete the proof of Lemma 2.2. Because $s_i(Y_i, D_i) = s_i(\min\{D_i, R_i, \tau_i\}, D_i)\mathbb{1}(Q_i \le q_i)$, if s_i is the Type-II service function, then

$$\mathbb{E}[s_i(Y_i, D_i) \mid \Lambda = \sigma] = \frac{1}{\mu_i} \mathbb{E}[\min\{D_i, R_i, \tau_i\} \mathbb{1}(Q_i \le q_i) \mid \Lambda = \sigma] = \frac{c_\sigma(i)x_i}{\mu_i} = c_\sigma(i)\beta_i$$

where $\beta_i = x_i/\mu_i$ for Type-II service by the definition of x_i and the derivation in (2.4). On the other hand, if s_i is the Type-III service function, then we express (2.9) as

$$c_{\sigma}(i)x_{i} = \int_{0}^{q_{i}} F_{i}^{-1}(q) \frac{\mathbb{E}[\min\{F_{i}^{-1}(q), R_{i}, \tau_{i}\} \mid \Lambda = \sigma]}{F_{i}^{-1}(q)} dq$$

$$\leq \frac{1}{q_{i}} \int_{0}^{q_{i}} F_{i}^{-1}(q) dq \int_{0}^{q_{i}} \frac{\mathbb{E}[\min\{F_{i}^{-1}(q), R_{i}, \tau_{i}\} \mid \Lambda = \sigma]}{F_{i}^{-1}(q)} dq$$

$$= \left(\frac{1}{q_{i}} \int_{0}^{q_{i}} F_{i}^{-1}(q) dq\right) \mathbb{E}[s_{i}(\min\{D_{i}, R_{i}, \tau_{i}\}, D_{i})\mathbb{1}(Q_{i} \leq q_{i}) \mid \Lambda = \sigma]$$
(2.10)

where we have applied the FKG inequality, noting that $F_i^{-1}(q)$ is non-decreasing in q while $\frac{\mathbb{E}[\min\{F_i^{-1}(q), R_i, \tau_i\}|\Lambda = \sigma]}{F_i^{-1}(q)}$ is non-increasing in q. If $q_i \leq F_i(1)$, then $x_i = \int_0^{q_i} F_i^{-1}(q) dq$ by definition and $q_i = \beta_i$ by the derivation in (2.5) for Type-III service. This would imply $c_{\sigma}(i)\beta_i \leq \mathbb{E}[s_i(Y_i, D_i) | \Lambda = \sigma]$, completing the proof. On the other hand, if $q_i > F_i(1)$, then the proof requires more complicated derivations of a similar nature. We show that (2.9) implies $\mathbb{E}[s_i(Y_i, D_i) | \Lambda = \sigma] \geq c_{\sigma}(i)\beta_i$ in the final case of Lemma 2.2, where s_i is the Type-III service function and $q_i > F_i(1)$. For brevity, we omit index i and the conditioning on $\Lambda = \sigma$. We derive

$$\mathbb{E}[s(Y,D)] = \int_{0}^{F(1)} \frac{\mathbb{E}[\min\{F^{-1}(q), R, \tau\}]}{F^{-1}(q)} dq + \int_{F(1)}^{q} \frac{\mathbb{E}[\min\{R, \tau\}]}{F^{-1}(q)} dq$$

$$\geq \frac{\int_{0}^{F(1)} \mathbb{E}[\min\{F^{-1}(q), R, \tau\}] dq}{\int_{0}^{F(1)} F^{-1}(q) dq} F(1) + \frac{(q - F(1))\mathbb{E}[\min\{R, \tau\}]}{q - F(1)} \int_{F(1)}^{q} \frac{1}{F^{-1}(q)} dq \quad (2.11)$$

by applying the FKG inequality on the first term, in the same way as in (2.10).

Our goal is to show that this is at least

$$\frac{\int_{0}^{F(1)} \mathbb{E}[\min\{F^{-1}(q), R, \tau\}] dq + (q - F(1)) \mathbb{E}[\min\{R, \tau\}]}{\int_{0}^{F(1)} F^{-1}(q) dq + q - F(1)} \left(F(1) + \int_{F(1)}^{q} \frac{1}{F^{-1}(q)} dq\right) \qquad (2.12)$$

$$= \frac{\int_{0}^{q} \mathbb{E}[\min\{F^{-1}(q), R, \tau\}] dq}{\int_{0}^{q} \min\{F^{-1}(q), 1\} dq} \int_{0}^{q} \min\{1, \frac{1}{F^{-1}(q)}\} dq$$

$$= \frac{c_{\sigma}(i) x_{i}}{x_{i}} \beta_{i}$$

which would complete the proof (the final equality applies (2.9), (2.7), and (2.5)).

To show that (2.11) is at least (2.12), we define the shorthand $a = \int_0^{F(1)} \mathbb{E}[\min\{F^{-1}(q), R, \tau\}] dq$, $b = \int_0^{F(1)} F^{-1}(q) dq$, $c = (q - F(1)) \mathbb{E}[\min\{R, \tau\}]$, d = q - F(1), x = F(1), and $y = \int_{F(1)}^q \frac{1}{F^{-1}(q)} dq$. Our goal is to prove that

$$\frac{a}{b}x + \frac{c}{d}y \ge \frac{a+c}{b+d}(x+y) \tag{2.13}$$

where $a, b, x \ge 0$ and c, d, y > 0 (because q > F(1)). We first handle the degenerate case b = 0, under which a = 0 and a/b = 1, which means (2.13) reduces to $x + \frac{c}{d}y \ge \frac{c}{d}(x+y)$ which is true because $c/d \le 1$. Now assuming b > 0, we see that (2.13) is equivalent to the following:

$$\frac{adx + bcy}{bd} \ge \frac{a+c}{b+d}(x+y)$$

$$(adx + bcy)(b+d) \ge (a+c)(x+y)bd$$

$$(da)(dx) + (bc)(by) \ge (da)(by) + (bc)(dx)$$

$$(2.14)$$

We prove the final inequality (2.14). Note that

$$d \cdot a = (q - F(1)) \int_0^{F(1)} \mathbb{E}[\min\{F^{-1}(q), R, \tau\}] dq = (q - F(1)) \int_0^{F(1)} \mathbb{E}[\min\{F^{-1}(q), \min\{R, \tau\}\}] dq$$
$$bc = \int_0^{F(1)} F^{-1}(q) dq(q - F(1)) \mathbb{E}[\min\{R, \tau\}] = (q - F(1)) \int_0^{F(1)} \mathbb{E}[F^{-1}(q) \cdot \min\{R, \tau\}] dq$$

so we have $da \ge bc \ge 0$ because the minimum of two numbers in [0,1] is greater than their product $(F^{-1}(q) \le 1 \text{ for } q \le F(1), \text{ and also } \min\{R, \tau\} \le 1)$. Meanwhile, note that

$$d \cdot x = (q - F(1))F(1)$$
$$by = \int_0^{F(1)} F^{-1}(q) dq \int_{F(1)}^q \frac{1}{F^{-1}(q)} dq$$

so we have $dx \ge by \ge 0$ because $F^{-1}(q) \le 1$ for $q \le F(1)$ and $F^{-1}(q) \ge 1$ for $q \ge F(1)$. By the rearrangement inequality, (2.14) holds indeed. This completes the proof.

Lemmas 2.1 and 2.2, in conjunction with the observations about convexity and the simple reduction for knapsack CRS, complete the proof of Theorem 1.1. We end with some remarks.

Remark 2. The reductions presented in this section hold under any method of generating the arrival order Λ (adversarial, uniformly random, etc.), as long as the method is the same in the online rationing problem and CRS. That being said, in this paper we will only apply the reductions for FB-CRS.

Remark 3. Implementing Algorithm 1 requires tracking the distribution of Rem, which can be approximated by sampling the history. We omit the details, as this assumption has been used in previous online CRS works (e.g., [EFGT22, MMG24]). We refer to [Ma18, MM24] for papers where this argument is formally spelled out.

3 Details of Single-unit FB-CRS Results

In the single-unit FB-CRS problem, an input is specified by (n, x), where $n \in \mathbb{N}$, and $x = (x_i)_{i=1}^n$ is a collection of probabilities. Recall that $\Pr[S_i = 1] = x_i$ for each $i \in [n]$.

In order to prove Theorems 1.2 and 1.3, we first write a mathematical program to characterize the optimal selection guarantee an FB-CRS can attain on (n, \mathbf{x}) . For each $i \in [n]$ and $\sigma \in \{f, b\}$, we introduce a variable $c_{\sigma}(i)$ corresponding to the probability that the FB-CRS accepts $i \in [n]$, given $S_i = 1$ and $\Lambda = \sigma$.

maximize
$$\min_{1 \le i \le n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2$$
(LP-SI)

subject to

$$c_{\sigma}(i) \le 1 - \sum_{j \le \sigma i} x_j \cdot c_{\sigma}(j) \qquad \qquad \forall i \in [n], \sigma \in \{\mathsf{f}, \mathsf{b}\}$$
(3.1)

$$c_{\sigma}(i) \ge 0 \qquad \forall i \in [n], \sigma \in \{\mathsf{f}, \mathsf{b}\} . \tag{3.2}$$

Here $\min_{1 \le i \le n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2$ corresponds to the selection guarantee achieved by the FB-CRS, and (3.1) says that if *i* is accepted assuming $\Lambda = \sigma$, then no previous element $j <_{\sigma} i$ could have been. Note that by introducing an additional variable $\beta \ge 0$, and the constraints $(c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2 \ge \beta$ for each $i \in [n]$, we can maximize for β , and reformulate LP-SI as a linear program (LP). Thus, LP-SI can be solved efficiently, and we denote the value of an optimal solution to LP-SI by LPOPT (n, \mathbf{x}) .

We now design an FB-CRS which we prove is $\text{LPOPT}(n, \boldsymbol{x})$ -selectable on (n, \boldsymbol{x}) . Afterwards, we show that this is best possible. That is, *no* FB-CRS is greater than $\text{LPOPT}(n, \boldsymbol{x})$ -selectable on (n, \boldsymbol{x}) .

Algorithm 2 Single-unit FB-CRS Input: elements [n] and $\boldsymbol{x} = (x_i)_{i=1}^n$ which satisfies $\sum_{i=1}^n x_i = \rho$. Output: at most element $i \in [n]$ with $S_i = 1$. 1: Compute an optimal solution of LP-SI to obtain $(c_f(i), c_b(i))_{i=1}^n$. 2: Observe realized permutation Λ and call it σ 3: for $i \in [n]$ arriving in increasing order of σ do 4: Draw $B_{\sigma}(i) \sim \text{Ber}\left(\frac{c_{\sigma}(i)}{1 - \sum_{j < \sigma^i} x_j \cdot c_{\sigma}(j)}\right)$ independently. 5: if $B_{\sigma}(i) \cdot S_i = 1$ and no element was previously accepted then 6: return i.

Remark 4. Algorithm 2 is well-defined, as $\frac{c_{\sigma}(i)}{1-\sum_{j<\sigma i} x_j \cdot c_{\sigma}(j)} \leq 1$ for each $\sigma \in \{\mathsf{f},\mathsf{b}\}$ and $i \in [n]$ by (3.1) of LP-SI.

Lemma 3.1 (proof in Appendix B.1). Algorithm 2 is LPOPT(n, x)-selectable on (n, x). Moreover, no FB-CRS is more than LPOPT(n, x)-selectable on (n, x).

3.1 FB-CRS Positive Result: Proving Theorem 1.2

Fix $\rho \ge 0$. Observe that to prove Theorem 1.2, without loss we can restrict to inputs (n, \mathbf{x}) with $\sum_{i=1}^{n} x_i = \rho$, and for which $\min_{i \in [n]} x_i > 0$. More, by Lemma 3.1, it suffices show that

$$LPOPT(n, \boldsymbol{x}) \ge \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho},$$
(3.3)

as then Algorithm 2 attains the selection guarantee claimed in Theorem 1.2. We prove (3.3) by constructing a feasible solution to LP-SI with value at least $\frac{\exp(\rho/2)}{1+\exp(\rho/2)\rho}$. To help us describe the solution, we first define a continuous function $\phi : [0, \rho] \to [0, 1]$. Specifically, for each $z \in [0, \rho]$,

$$\phi(z) := \begin{cases} \frac{2e^{\rho/2} - e^z}{1 + e^{\rho/2}\rho} & \text{if } z \le \rho/2, \\ \frac{e^{\rho-z}}{1 + e^{\rho/2}\rho} & \text{if } \rho/2 < z \le \rho. \end{cases}$$
(3.4)

Note that (3.4) generalizes (1.9) from Subsection 1.4 to an arbitrary value of $\rho \ge 0$.

Proposition 3.2 (Proof in Appendix B.2). Function $\phi : [0, \rho] \rightarrow [0, 1]$ defined in (3.4) satisfies the following:

- 1. ϕ is continuous and decreasing on $[0, \rho]$.
- 2. For each $z \in [0, \rho]$,

$$\frac{\phi(z) + \phi(\rho - z)}{2} = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho};$$
(3.5)

$$\phi(z) \le 1 - \int_0^z \phi(\tau) d\tau. \tag{3.6}$$

Remark 5. Properties (3.5) and (3.6) correspond to the objective and constraints of LP-SI for an input with $\max_{1 \le i \le n} x_i \to 0$. Thus, we can interpret $(\phi(z), \phi(\rho - z))_{0 \le z \le \rho}$ as a limiting solution of LP-SI as $\max_{1 \le i \le n} x_i \to 0$. The first property (1) is a technical assumption to help us verify (3.3).

For each $i \in [n]$ and $\sigma \in \{f, b\}$, let us now define $x_{\sigma}(i) := \sum_{j < \sigma i} x_j$ where $x_f(1) = x_b(n) := 0$ for convenience. Using ϕ , and recalling that $\sum_{i=1}^n x_i = \rho$, we define $(c_f(i), c_b(i))_{i=1}^n$ as follows:

$$c_{\mathbf{f}}(i) := \int_{x_{\mathbf{f}}(i)}^{x_{\mathbf{f}}(i)+x_i} \frac{\phi(\tau)}{x_i} d\tau, \text{ and } c_{\mathbf{b}}(i) := \int_{x_{\mathbf{b}}(i)}^{x_{\mathbf{b}}(i)+x_i} \frac{\phi(\tau)}{x_i} d\tau .$$
(3.7)

Here $= c_{\sigma}(i)$ is the average value of the function ϕ on the interval $[x_{\sigma}(i), x_{\sigma}(i) + x_i]$. As such, $c_{\sigma}(i)$ agrees exactly with ϕ for inputs with $\max_{1 \le i \le n} x_i \to 0$, and is a decreasing function on $[0, \rho]$. Thus, the *further* an element *i* is in the order specified by $\sigma \in \{f, b\}$, the *smaller* the value of $c_{\sigma}(i)$.

Lemma 3.3. Fix an input (n, \mathbf{x}) with $\sum_{i=1}^{n} x_i = \rho$. Then, $(c_f(i), c_b(i))_{i=1}^n$ defined in (3.7) is a feasible solution to LP-SI with

$$LPOPT(n, \boldsymbol{x}) \ge \min_{1 \le i \le n} \frac{c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i)}{2} = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho}.$$
 (3.8)

Proof of Lemma 3.3. Recall that $\sum_{i=1}^{n} x_i = \rho$, and $\min_{i \in [n]} x_i > 0$. To verify the left-most inequality of (3.8), we first argue that $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ is a feasible solution to LP-SI. In order to see this, observe that

$$\sum_{j < i} c_{\mathsf{f}}(j) x_j = \sum_{j < i} \int_{x_{\mathsf{f}}(j)}^{x_{\mathsf{f}}(j) + x_j} \phi(\tau) d\tau = \int_0^{x_{\mathsf{f}}(i)} \phi(\tau) d\tau.$$
(3.9)

On the other hand, since ϕ is a decreasing function (by Proposition 3.2),

$$c_{\mathsf{f}}(i) = \int_{x_{\mathsf{f}}(i)}^{x_{\mathsf{f}}(i)+x_i} \frac{\phi(\tau)d\tau}{x_i} \le \phi(x_{\mathsf{f}}(i)).$$
(3.10)

By applying (3.9) and (3.10), we get that $c_{\mathsf{f}}(i) + \sum_{j < i} c_{\mathsf{f}}(j) x_j \leq \phi(x_{\mathsf{f}}(i)) + \int_0^{x_{\mathsf{f}}(i)} \phi(\tau) d\tau \leq 1$, where the final inequality uses (3.6) of Proposition 3.2. Similarly, by the definition of $(c_{\mathsf{b}}(i))_{i=1}^n$,

$$c_{\mathsf{b}}(i) + \sum_{j>i} c_{\mathsf{b}}(j) x_j \le \phi(x_{\mathsf{b}}(i)) + \int_0^{x_{\mathsf{b}}(i)} \phi(\tau) d\tau \le 1.$$

Thus, $(c_{\rm f}(i), c_{\rm b}(i))_{i=1}^n$ is feasible, and so the leftmost inequality of (3.8) is established. It remains to verify the rightmost inequality of (3.8). Observe that for any $i \in [n]$, we have that $x_{\rm b}(i) + x_i = \rho - x_{\rm f}(i)$. Thus,

$$\frac{c_{f}(i) + c_{b}(i)}{2} = \int_{x_{f}(i)}^{x_{f}(i) + x_{i}} \frac{\phi(\tau)}{2x_{i}} d\tau + \int_{x_{b}(i)}^{x_{b}(i) + x_{i}} \frac{\phi(\tau)}{2x_{i}} d\tau$$
$$= \int_{x_{f}(i)}^{x_{f}(i) + x_{i}} \frac{\phi(\tau)}{2x_{i}} d\tau + \int_{\rho - x_{f}(i) - x_{i}}^{\rho - x_{f}(i)} \frac{\phi(\tau)}{2x_{i}} d\tau$$
$$= \frac{1}{x_{i}} \left(\int_{x_{f}(i)}^{x_{f}(i) + x_{i}} \frac{\phi(\tau) + \phi(\rho - \tau)}{2} d\tau \right) = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho},$$

where the third equality applies a change of variables, and the last equality applies (3.5) of Proposition 3.2. Thus, $\min_{i \in [n]} \frac{c_{f}(i) + c_{b}(i)}{2} = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho}$, and so the proof is complete.

Lemma 3.3 implies (3.3), and so due to the discussion at the beginning of the section, this completes the proof of Theorem 1.2.

3.2 Single-unit FB-CRS Hardness Result: Proving Theorem 1.3

To prove Theorem 1.3, we fix an arbitrary $\rho \ge 0$, and consider the input (N, \boldsymbol{x}) with $\sum_{i=1}^{N} x_i = \rho$, and $x_i := \rho/N$ for all $i \in [N]$. For convenience, we assume that N is odd; that is N = 2n + 1 for some $n \ge 0$. Our goal is to prove the following.

Theorem 3.4. Fix $\rho \ge 0$, and N = 2n + 1 for $n \ge 0$. If $x_i = \rho/N$ for all $i \in [N]$, then

$$LPOPT(N, \boldsymbol{x}) \le \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho} + \frac{\rho + 2}{N} .$$

Due to Lemma 3.1, we can then take $N \to \infty$ to establish Theorem 1.3. Thus, the remainder of the section is focused on proving Theorem 3.4. In order to prove it, we first take the dual of LP-SI on the input (N, x). We reformulate it in an equivalent way that is more convenient for our purposes (see Appendix B.3 for details):

minimize

$$\sum_{i=1}^{N} \frac{(y_{\mathsf{f}}(i) + y_{\mathsf{b}}(i))}{N}$$
(dual-LP-SI)

subject to

$$y_{\sigma}(i) + \sum_{j > \sigma i} \frac{\rho \cdot y_{\sigma}(j)}{N} - \frac{\xi(i)}{2} \ge 0 \qquad \forall i \in [N], \sigma \in \{\mathsf{f}, \mathsf{b}\}$$
(3.11)

$$\sum_{i=1}^{n} \frac{\xi(i)}{N} \ge 1 \tag{3.12}$$

$$\xi(i), y_{\mathsf{f}}(i), y_{\mathsf{b}}(i) \ge 0 \qquad \qquad \forall i \in [N].$$
(3.13)

Our goal is to construct a feasible solution $(\xi(i), y_{\mathsf{f}}(i), y_{\mathsf{b}}(i))_{i=1}^{N}$ to dual-LP-SI of value at most $\alpha_0 + \frac{\rho+2}{N}$, where $\alpha_0 := \frac{\exp(\rho/2)}{1+\exp(\rho/2)\rho}$. By weak duality, this will imply Theorem 3.4. We begin by defining $(\xi(i))_{i=1}^{N}$, which is constant except at element n + 1. Specifically,

$$\xi(i) = \begin{cases} \rho \cdot \alpha_0 & \text{if } i \neq n+1, \\ \left(1 - \rho \cdot \alpha_0 \frac{N-1}{N}\right) \cdot N & \text{if } i = n+1. \end{cases}$$
(3.14)

In order to state $(y_{f}(i), y_{b}(i))_{i=1}^{N}$, we first define a function $\gamma : [\rho/2, \rho] \to [0, 1]$, where for each $z \in [\rho/2, \rho],$

$$\gamma(z) := \frac{\rho \exp(z - \rho/2)}{2(1 + \exp(\rho/2)\rho)}.$$
(3.15)

The solution $(y_f(i))_{i=1}^N$ is then identically 0 for i < n+1, takes on value $(1+\xi(n+1))/2$ at i = n+1, and is otherwise $\gamma(\rho \cdot i/N)$. Finally, $y_{\mathbf{b}}(i) := y_{\mathbf{f}}(N - (i-1))$ for each $i \in [N]$. To summarize,

$$y_{\mathsf{f}}(i) := \begin{cases} 0 & \text{if } i < n+1, \\ \frac{\xi(n+1)}{2} + \frac{1}{2} & \text{if } i = n+1, \\ \gamma(\rho i/N) & \text{if } n+1 < i \le N. \end{cases} \\ \begin{pmatrix} \gamma(\rho - \rho(i-1)/N) & \text{if } i < n+1, \\ \frac{\xi(n+1)}{2} + \frac{1}{2} & \text{if } i = n+1, \\ 0 & \text{if } n+1 < i \le N. \end{cases}$$
(3.16)

Lemma 3.5. Fix $\rho \ge 0$, N = 2n + 1 for $n \ge 0$, and set $x_i = \rho/N$ for all $i \in [N]$. Then, $(\xi(i), y_{\mathsf{f}}(i), y_{\mathsf{b}}(i))_{i=1}^{N}$ as defined in (3.14) and (3.16) is a feasible solution to dual-LP-SI for which

$$\sum_{i=1}^{N} \frac{y_{\mathsf{f}}(i) + y_{\mathsf{b}}(i)}{N} \le \alpha_0 + \frac{\rho + 2}{N}.$$

In order to prove Lemma 3.5, we need the following properties of γ :

Proposition 3.6 (proof in Appendix B.4). Function $\gamma : [\rho/2, \rho] \to [0, 1]$ defined in (3.15) satisfies the following:

- 1. γ is 1-Lipschitz and increasing.
- 2. For each $z \in [\rho/2, \rho]$,

$$\gamma(z) + \int_{z}^{\rho} \gamma(\tau) d\tau = \frac{\rho \cdot \alpha_0}{2}.$$
(3.17)

Remark 6. Since $\xi(i) = \frac{\rho \cdot \alpha_0}{2}$, except for at i = n + 1, property (3.17) corresponds to constraint (3.11) as $N \to \infty$. Since $(\gamma(z))_{\rho/2 < z \le \rho}$ (respectively, $(\gamma(\rho - z))_{0 \le z < \rho/2}$) is the limit of $(y_{\mathsf{f}}(i))_{i > n+1}$ (respectively, $(y_{\mathsf{b}}(i))_{i < n+1}$), this suggests (3.11) holds for i > n + 1 (respectively, i < n + 1). That being said, at i = n + 1, $y_{\mathsf{f}}(n + 1) = y_{\mathsf{b}}(n + 1) \to (1 - \alpha_0)N/2$ as $N \to \infty$, yet $\gamma(\rho/2)$ is a constant.

Proof of Lemma 3.5. We begin by verifying the feasibility of the solution to dual-LP-SI. Clearly, $\sum_{i=1}^{N} \frac{\xi(i)}{N} = \frac{(N-1)\rho\alpha_0}{N} + (1 - \rho\alpha_0 \frac{N-1}{N}) = 1$, so (3.12) is satisfied (at equality). We next verify (3.11) for $\sigma = f$. If i = n + 1, then this is immediate, since $y_f(n+1) \ge \xi(n+1)/2$. For $i \ge n+2$, recall that $y_f(i) := \gamma(\rho i/N)$. Now, since γ is an increasing function,

$$\sum_{j=i}^{N} \frac{y_{\mathsf{f}}(j)}{N} = \sum_{j=i}^{N} \frac{\rho \cdot \gamma(\rho j/N)}{N} \ge \int_{\rho(i-1)/N}^{\rho} \gamma(\tau) d\tau, \qquad (3.18)$$

where we've interpreted $\sum_{j=i}^{N} \frac{\rho \cdot \gamma(\rho j/N)}{N}$ as a right endpoint Riemann sum. By applying (3.18),

$$y_{f}(i) + \sum_{j=i+1}^{N} \frac{\rho \cdot y_{f}(j)}{N} \ge \gamma(\rho i/N) + \int_{\rho i/N}^{\rho} \gamma(\tau) d\tau = \frac{\rho \alpha_{0}}{2} = \frac{\xi(i)}{2},$$

where the final equality applies (3.17) of Proposition 3.6, and the definition of $\xi(i)$ for i > n + 1. Thus, (3.11) is satisfied for i > n + 1. Finally, for i < n, we know that $y_{f}(i) = 0$, and $\xi(i) = \rho \alpha_{0}$. On the other hand, observe that α_{0} satisfies

$$1 - \alpha_0 \rho = \alpha_0 e^{-\rho/2}, \tag{3.19}$$

and so $\frac{\rho(1-\alpha_0\rho)}{2} = \frac{\alpha_0\rho}{2}e^{-\rho/2} = \gamma(\rho/2)$. More, since $\frac{y_f(n+1)}{N} \ge \frac{1-\alpha_0\rho}{2} + \frac{1}{2N}$, we get that

$$\frac{\rho \cdot y_{\mathsf{f}}(n+1)}{N} \ge \gamma(\rho/2) + \frac{\rho}{2N} \ge \gamma(\rho/2) + \int_{\rho/2}^{\rho(n+1)/N} \gamma(\tau) d\tau, \tag{3.20}$$

where the final inequality uses that $\gamma(\tau) \leq 1$ on $[\rho/2, \rho(n+1)/N]$. By applying (3.20) followed by (3.18),

$$\begin{aligned} \frac{\rho \cdot y_{\mathsf{f}}(n+1)}{N} + \sum_{j=n+2}^{N} \frac{\rho \cdot y_{\mathsf{f}}(j)}{N} &\geq \gamma(\rho/2) + \int_{\rho/2}^{\rho(n+1)/N} \gamma(\tau) d\tau + \sum_{j=n+2}^{N} \frac{\rho y_{\mathsf{f}}(j)}{N} \\ &\geq \gamma(\rho/2) + \int_{\rho/2}^{\rho(n+1)/N} \gamma(\tau) d\tau + \int_{\frac{\rho(n+1)}{N}}^{\rho} \gamma(\tau) d\tau \\ &= \gamma(\rho/2) + \int_{\frac{\rho}{2}}^{\rho} \gamma(\tau) d\tau = \frac{\rho \alpha_0}{2} = \frac{\xi(i)}{2}, \end{aligned}$$

where the last two equalities use Proposition 3.6 and the definition of $\xi(i)$ for i < n. Thus, (3.11) holds for i < n. Due to the symmetry of $(y_{\mathsf{b}}(i))_{i=1}^{N}$, we know that (3.11) also holds for $\sigma = \mathsf{b}$.

It remains to bound the value of the feasible solution. Now, using that $y_{f}(i) = y_{b}(N - (i - 1))$ for $i \in [N]$, and $y_{f}(i) = 0$ for i < n,

$$\sum_{i=1}^{N} \frac{(y_{\mathsf{f}}(i) + y_{\mathsf{b}}(i))}{N} = \frac{2y_{\mathsf{f}}(n+1)}{N} + 2\sum_{i=n+2}^{N} \frac{y_{\mathsf{f}}(i)}{N}$$

$$= \left(1 - \rho \cdot \alpha_0 \frac{N-1}{N}\right) + \frac{1}{N} + \frac{2}{\rho} \sum_{i=n+2}^{N} \frac{\rho \cdot y_{\mathbf{f}}(i)}{N}, \qquad (3.21)$$

where the second equality uses $y_{f}(n+1) = y_{b}(n+1) = \left(1 - \rho \cdot \alpha_{0} \frac{(N-1)}{N}\right) \frac{N}{2} + \frac{1}{2}$. Since γ is 1-Lipschitz by Proposition 3.6,

$$\sum_{i=n+2}^{N} \frac{\rho \cdot y_{\mathbf{f}}(i)}{N} = \sum_{i=n+2}^{N} \frac{\rho \cdot \gamma(\rho(i-1)/N)}{N} + \sum_{i=n+2}^{N} \left(\frac{\rho \cdot \gamma(\rho i/N)}{N} - \frac{\rho \cdot \gamma(\rho(i-1)/N)}{N} \right)$$
$$\leq \sum_{i=n+2}^{N} \frac{\rho \cdot \gamma(\rho(i-1)/N)}{N} + \frac{\rho}{N}$$
$$\leq \int_{\rho/2}^{s} \gamma(\tau) d\tau + \frac{\rho}{N}, \tag{3.22}$$

where the final inequality uses that γ is increasing, and interprets $\sum_{i=n+2}^{N} \frac{\rho \cdot \gamma(\rho(i-1)/N)}{N}$ as a left endpoint Riemann sum. Combining (3.21) with (3.22), and using $\frac{2}{\rho} \int_{\rho/2}^{\rho} \gamma(\tau) d\tau = \alpha_0 - \alpha_0 e^{-\rho/2}$ due to Proposition 3.6,

$$\sum_{i=1}^{N} \frac{(y_{\mathbf{f}}(i) + y_{\mathbf{b}}(i))}{N} \le 1 - \rho \cdot \alpha_0 \frac{N-1}{N} + \frac{2}{\rho} \int_{\rho/2}^{\rho} \gamma(\tau) d\tau + \frac{2}{N}$$
$$= 1 - \rho \cdot \alpha_0 \frac{N-1}{N} + \alpha_0 - \alpha_0 e^{-\rho/2} + \frac{2}{N}$$
$$= \alpha_0 + (1 - \rho \cdot \alpha_0 - \alpha_0 e^{-\rho/2}) + \frac{\rho \cdot \alpha_0 + 2}{N} \le \alpha_0 + \frac{\rho + 2}{N},$$

where the final inequality uses (3.19) and that $\alpha_0 \leq 1$. The proof is thus complete.

4 Details of Knapsack FB-CRS Results

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4.1 Knapsack FB-CRS Positive Result: Proving Theorem 1.4

Given a knapsack input $(n, (F_i)_{i=1}^n)$, recall that F_i is a distribution on $[0, 1] \cup \{\infty\}$, and $S_i \sim F_i$ is the random size of element *i*. If $\mu_i := \mathbb{E}[S_i \cdot \mathbb{1}(S_i < \infty)]$ for each $i \in [n]$, then we may assume that $\sum_{i=1}^n \mu_i = 1$ and $\mu_i > 0$ for each $i \in [n]$.

Our high level approach to proving Theorem 1.4 is closely related to how we designed our singleunit FB-CRS from Section 3. Specifically, let $(c_{\rm f}(i), c_{\rm b}(i))_{i=1}^n$ be a collection of probabilities. Our goal is to define an FB-CRS with respect to $(c_{\rm f}(i), c_{\rm b}(i))_{i=1}^n$, such that if A_i denotes the indicator random variable for the event that *i* is accepted, then $\Pr[A_i | S_i = s, \Lambda = \sigma] = c_{\sigma}(i)$, for each $\sigma \in \{f, b\}, i \in [n]$, and $s \in [0, 1]$. By averaging over Λ , this will imply a selection guarantee of $\min_{1 \le i \le n} (c_{\rm f}(i) + c_{\rm b}(i))/2$ is attainable. Unlike the single-unit setting, our probabilities will *not* come from an LP. However, we still must ensure that $(c_{\rm f}(i), c_{\rm b}(i))_{i=1}^n$ satisfy certain inequalities. For convenience, we denote $i_1 := \sigma^{-1}(1) \in \{1, n\}$ to be the first element with respect to σ :

Definition 5. We refer to probabilities $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^n$ as *feasible* for $(n, (F_i)_{i=1}^n)$ provided:

1. For each $1 \le i \le n-1$, $c_{f}(i+1) \le c_{f}(i)$, and $c_{b}(i) \le c_{b}(i+1)$.

2. For each $i \in [n]$, and $\sigma \in \{f, b\}$:

$$c_{\sigma}(i) \le 1 - c_{\sigma}(i_1) - \sum_{j < \sigma^i} c_{\sigma}(j) \cdot \mu_j; \tag{4.1}$$

$$c_{\sigma}(i) \le 1 - 2\sum_{j < \sigma i} c_{\sigma}(j) \cdot \mu_j - c_{\sigma}(i_1) \cdot \exp\left(\frac{-2}{c_{\sigma}(i_1)}\sum_{j < \sigma i} c_{\sigma}(j) \cdot \mu_j\right).$$

$$(4.2)$$

We now define our FB-CRS (Algorithm 3, formally written on the next page) with respect to an arbitrary choice of feasible probabilities. Afterwards, we shall construct a choice of feasible probabilities that implies the claimed selection guarantee of Theorem 1.4. This second step is analogous to the approach taken in Subsection 3.1.

Let us condition on $\Lambda = \sigma$ for $\sigma \in \{f, b\}$ and $S_i = s_i \in [0, 1]$ for $i \in [n]$. Our FB-CRS is defined recursively with respect to the permutation σ . That is, assuming we've defined the algorithm for all elements $j <_{\sigma} i$, we extend its definition to the current arrival i. In order to do so, denote $\mathcal{T}_{\sigma}(i) := \sum_{j <_{\sigma} i} S_j \cdot A_j$, where $\mathcal{T}_{f}(1) := 0$ and $\mathcal{T}_{b}(n) := 0$ (recall that A_j is an indicator random variable for the event that j is accepted). Observe that since $\mathcal{T}_{\sigma}(i)$ depends on the elements $j <_{\sigma} i$, the probabilities $\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]$ and $\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]$ are well-defined. Further note that $\Pr[\mathcal{T}_{\sigma}(i_1) = 0 \mid \Lambda = \sigma] = 1$ and $\Pr[0 < \mathcal{T}_{\sigma}(i_1) \le 1 - s_{i_1} \mid \Lambda = \sigma] = 0$.

Our FB-CRS will again use a random bit $B_{\sigma}(i)$ to decide whether to accept *i*; however this bit will now depend on the value of $\mathcal{T}_{\sigma}(i)$ in the current execution of the FB-CRS. Specifically, if $0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i$, then

$$B_{\sigma}(i) \sim \operatorname{Ber}\left(\min\left(1, \frac{c_{\sigma}(i)}{\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]}\right)\right).$$
(4.3)

Else if $\mathcal{T}_{\sigma}(i) = 0$ and $c_{\sigma}(i) > \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]$, then

$$B_{\sigma}(i) \sim \operatorname{Ber}\left(\min\left(1, \frac{c_{\sigma}(i) - \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]}{\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]}\right)\right).$$
(4.4)

Otherwise, we pass on i (i.e., $B_{\sigma}(i) = 0$).

As we shall argue in Lemma 4.2, if a certain induction hypothesis holds, then the minimum in (4.4) is unnecessary, and so (4.3) and (4.4) are calibrated to ensure that

$$\Pr[A_i \mid S_i = s_i, \Lambda = \sigma] = c_{\sigma}(i). \tag{4.5}$$

Roughly speaking, we maintain this induction hypothesis for the next arriving element by prioritizing the acceptance of *i* on executions when $\mathcal{T}_{\sigma}(i) > 0$, and $\mathcal{T}_{\sigma}(i) + s_i \leq 1$. More precisely, if $c_{\sigma}(i) \leq \Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$, then we accept *i* only if $0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i$. On the other hand, if $c_{\sigma}(i) > \Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$ then we greedily accept *i* when $0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i$, and otherwise accept it when $\mathcal{T}_{\sigma}(i) = 0$ only as much as needed to ensure that (4.5) holds.

Remark 7. In Algorithm 3, computing $\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]$ and $\Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$ exactly requires tracking exponentially many scenarios. However, this can be avoided by using random sampling in a similar way as discussed in Remark 3. One could also use a discretization argument for knapsack [JMZ22].

Theorem 4.1. Suppose Algorithm 3 is defined using feasible selection values $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ for $(n, (F_i)_{i=1}^{n}$. Then, for each $\sigma \in \{\mathsf{f}, \mathsf{b}\}, i \in [n]$ and $s_i \in [0, 1]$, $\Pr[A_i \mid S_i = s_i, \Lambda = \sigma] = c_{\sigma}(i)$.

Algorithm 3 Knapsack FB-CRS

Input: knapsack input $(n, (F_i)_{i=1}^n)$ which satisfies $\sum_{i=1}^n \mu_i \leq 1$. **Output:** a subset of elements $\mathcal{I} \subseteq [n]$ with $\sum_{i \in \mathcal{I}} S_i \leq 1$. 1: $\mathcal{I} \leftarrow \emptyset$. 2: Observe realized permutation Λ and call it σ 3: for $i \in [n]$ arriving in increasing order of σ with $S_i = s_i \in [0, 1]$ do Set $\mathcal{T}_{\sigma}(i) := \sum_{j < \sigma i} S_j \cdot A_j$. Based on the algorithm's execution on the elements $j <_{\sigma} i$, compute $\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]$ 4: 5: and $\Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma].$ if $0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i$ then 6: Draw $B_{\sigma}(i) \sim \operatorname{Ber}\left(\min\left(1, \frac{c_{\sigma}(i)}{\operatorname{Pr}[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]}\right)\right)$. else if $\mathcal{T}_{\sigma}(i) = 0$ and $c_{\sigma}(i) > \operatorname{Pr}[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$ then Draw $B_{\sigma}(i) \sim \operatorname{Ber}\left(\min\left(1, \frac{c_{\sigma}(i) - \operatorname{Pr}[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]}{\operatorname{Pr}[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]}\right)\right)$. 7: 8: 9: else10: 11: $B_{\sigma}(i) = 0.$ if $B_{\sigma}(i) = 1$ then $I \leftarrow I \cup \{i\}$. \triangleright If $B_{\sigma}(i) = 1$, then $s_i + \mathcal{T}_{\sigma}(i) \leq 1$. 12:13: return \mathcal{I} .

In order to prove Theorem 4.1, we define the following induction hypothesis:

1. For each $\sigma \in {f, b}$ and $i \in [n]$, for all $0 < b \le 1/2$,

$$\frac{\Pr[0 < \mathcal{T}_{\sigma}(i) \le b \mid \Lambda = \sigma]}{c_{\sigma}(i_1)} \le \exp\left(-\frac{\Pr[b < \mathcal{T}_{\sigma}(i) \le 1 - b \mid \Lambda = \sigma]}{c_{\sigma}(i_1)}\right);$$
(4.6)

2. For each $\sigma \in \{f, b\}$ and $i \in [n]$,

$$\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma] \ge c_{\sigma}(i). \tag{4.7}$$

The following lemma shows that if element i satisfies (4.7), then the minimum of the Bernoulli parameter in line 9 of Algorithm 3 is unnecessary, and so Algorithm 3 accepts i as specified in Theorem 4.1. This would complete the proof of Theorem 4.1.

Lemma 4.2. Fix $\sigma \in \{f, b\}$. If (4.7) holds for $i \in [n]$, then for each $s_i \in [0, 1]$,

1.

$$\frac{c_{\sigma}(i) - \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]}{\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]} \le 1;$$
(4.8)

2.

$$\Pr[A_i \mid S_i = s_i, \Lambda = \sigma] = c_{\sigma}(i).$$
(4.9)

Proof of Lemma 4.2. Observe first that,

$$\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma] + \Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma] = \Pr[\mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]$$
$$\ge \Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]$$

 $\geq c_{\sigma}(i),$

where the last inequality follows by the assumption (4.7) for *i*. Thus, we can now subtract the term $\Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$ from both sides, and then divide by $\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]$ to get (4.8).

Let us now implicitly condition on $\Lambda = \sigma$ and $S_i = s_i$ for the remainder of the proof. Observe then that A_i occurs if and only if $\{0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i\} \cap \{B_{\sigma}(i) = 1\}$ or $\{\mathcal{T}_{\sigma}(i) = 0\} \cap \{B_{\sigma}(i) = 1\}$. Since the latter are disjoint events,

$$\Pr[A_i \mid S_i = s_i, \Lambda = \sigma] = \Pr[\{0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i\} \cap \{B_{\sigma}(i) = 1\} \mid S_i = s_i, \Lambda = \sigma] + \Pr[\{\mathcal{T}_{\sigma}(i) = 0\} \cap \{B_{\sigma}(i) = 1\} \mid S_i = s_i, \Lambda = \sigma].$$
(4.10)

In order to simplify (4.10), we first consider the case when $c_{\sigma}(i) \leq \Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 - s_i \mid \Lambda = \sigma]$. Observe then that by the definition of $B_{\sigma}(i)$, for each $d_i \in [0, 1 - s_i]$,

$$\Pr[B_{\sigma}(i) = 1 \mid \Lambda = \sigma, S_i = s_i, \mathcal{T}_{\sigma}(i) = d_i] = \begin{cases} \frac{c_{\sigma}(i)}{\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]} & \text{if } 0 < d_i \le 1 - s_i. \\ 0 & \text{if } d_i = 0. \end{cases}$$

$$(4.11)$$

Thus, by applying (4.11) to the RHS of (4.10), we can write $\Pr[A_i \mid S_i = s_i, \Lambda = \sigma]$ as:

$$\begin{aligned} &\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma, S_i = s_i] \cdot \Pr[B_{\sigma}(i) = 1 \mid \Lambda = \sigma, S_i = s_i, 0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i] \\ &= \frac{c_{\sigma}(i)}{\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]} \cdot \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma] = c_{\sigma}(i), \end{aligned}$$

and so (4.9) holds. It remains to consider the case when $c_{\sigma}(i) > \Pr[0 \leq \mathcal{T}_{\sigma}(i) > 1 - s_i \mid \Lambda = \sigma]$. Since we've already proven (4.8), by the definition of $B_{\sigma}(i)$ we know that for each $d_i \in [0, 1 - s_i]$,

$$\Pr[B_{\sigma}(i) = 1 \mid \Lambda = \sigma, S_i = s_i, \mathcal{T}_{\sigma}(i) = d_i] = \begin{cases} 1 & \text{if } 0 < d_i \le 1 - s_i \\ \frac{c_{\sigma}(i) - \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]}{\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]} & \text{if } d_i = 0. \end{cases}$$

$$(4.12)$$

By applying (4.12) to the RHS side of (4.10), we can write $\Pr[A_i \mid S_i = s_i, \Lambda = \sigma]$ as

$$\Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma, S_i = s_i] + \left(\frac{c_{\sigma}(i) - \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 - s_i \mid \Lambda = \sigma]}{\Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma]}\right) \cdot \Pr[\mathcal{T}_{\sigma}(i) = 0 \mid \Lambda = \sigma] = c_{\sigma}(i),$$

and so (4.9) holds. The proof is thus complete.

For the induction hypothesis, we now discuss how (4.6) relates to (4.7). By Lemma 4.2, to establish the guarantee of Theorem 4.1, it suffices to prove (4.7) or equivalently $\Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 \mid \Lambda = \sigma] \leq 1 - c_{\sigma}(i)$. On the other hand, as we shall see in our inductive argument, we can apply Lemma 4.2 to write $\mathbb{E}[\mathcal{T}_{\sigma}(i) \mid \Lambda = \sigma] = \sum_{j < \sigma^{i}} c_{\sigma}(j)\mu_{j}$. By rewriting $\mathbb{E}[\mathcal{T}_{\sigma}(i) \mid \Lambda = \sigma]$ using integration by parts, we get the following:

$$\sum_{j < \sigma i} c_{\sigma}(j)\mu_j = \mathbb{E}[\mathcal{T}_{\sigma}(i) \mid \Lambda = \sigma] = \Pr[0 < \mathcal{T}_{\sigma}(i) \le 1 \mid \Lambda = \sigma] - \int_0^1 \Pr[0 < \mathcal{T}_{\sigma}(i) \le \tau \mid \Lambda = \sigma] d\tau.$$
(4.13)

We shall use (4.6) to upper-bound the integral in (4.13), which combined with Definition 5, will allow us to prove $\Pr[0 < \mathcal{T}_{\sigma}(i) \leq 1 \mid \Lambda = \sigma] \leq 1 - c_{\sigma}(i)$ as desired. In this way, we can roughly interpret (4.6) as an *anti-concentration* inequality: it controls the amount of mass $\mathcal{T}_{\sigma}(i)$ can have away from 0. Our formal induction argument establishing (4.6) and (4.7) is *deferred to Appendix C.1*.

Having verified (4.7) and (4.6) for all $i \in [n]$ and $\sigma \in \{f, b\}$, Lemma 4.2 immediately implies Theorem 4.1. We now construct a specific choice of feasible $(c_f(i), c_b(i))_{i=1}^n$ which implies the selection guarantee claimed in Theorem 1.4. As in the single-unit case, we first describe a continuous function $\phi : [0, 1] \rightarrow [0, 1]$ to help us describe our solution. Specifically, for each $z \in [0, 1]$,

$$\phi(z) := \frac{4}{9} - \frac{2z}{9}.\tag{4.14}$$

Proposition 4.3 (proof in Appendix C.2). Function ϕ defined in (4.14) satisfies the following:

- 1. ϕ is decreasing and continuous on [0, 1].
- 2. For each $z \in [0, 1]$:

$$\frac{\phi(z) + \phi(1-z)}{2} = \frac{1}{3} \tag{4.15}$$

$$\phi(z) \le 1 - \phi(0) - \int_0^z \phi(\tau) d\tau$$
 (4.16)

$$\phi(z) \le 1 - 2\int_0^z \phi(\tau)d\tau - \phi(0) \cdot \exp\left(\frac{-2}{\phi(0)}\int_0^z \phi(\tau)d\tau\right) .$$
(4.17)

Remark 8. Properties (4.16) and (4.17) correspond to (4.1) and (4.2) of Definition 5 for an input with $\max_{1 \le i \le n} \mu_i \to 0$. Thus, we can interpret $(\phi(z), \phi(1-z))_{0 \le z \le 1}$ as a limiting solution of Definition 5 as $\max_{1 \le i \le n} \mu_i \to 0$.

For each $i \in [n]$ and $\sigma \in \{f, b\}$, define $\mu_{\sigma}(i) := \sum_{j < \sigma^i} \mu_j$ where $\mu_f(1) = \mu_b(n) := 0$ for convenience. Using ϕ , and recalling that $\sum_{i=1}^n \mu_i = 1$, we define $(c_f(i), c_b(i))_{i=1}^n$ in the following way:

$$c_{\mathsf{f}}(i) := \int_{\mu_{\mathsf{f}}(i)}^{\mu_{\mathsf{f}}(i)+\mu_{i}} \frac{\phi(\tau)}{\mu_{i}} d\tau, \text{ and } c_{\mathsf{b}}(i) := \int_{\mu_{\mathsf{b}}(i)}^{\mu_{\mathsf{b}}(i)+\mu_{i}} \frac{\phi(\tau)}{\mu_{i}} d\tau.$$
(4.18)

Here we can interpret $c_{\sigma}(i)$ as the average value of the function ϕ on the interval $[\mu_{\sigma}(i), \mu_{\sigma}(i) + \mu_i]$. As such, $c_{\sigma}(i)$ agrees exactly with ϕ for inputs with $\max_{1 \le i \le n} \mu_i \to 0$. Note that ϕ is a decreasing function on [0, 1]. Thus, the *further* an element *i* is in the order specified by $\sigma \in \{f, b\}$, the *smaller* the value of $c_{\sigma}(i)$. The next lemma proceeds similarly to Lemma 3.3 from Subsection 3.1.

Lemma 4.4 (proof in Appendix C.3). Fix an input $(n, (F_i)_{i=1}^n)$ with $\sum_{i=1}^n \mu_i = 1$ and $\min_{i \in [n]} \mu_i > 0$. Then, $(c_f(i), c_b(i))_{i=1}^n$ as defined in (4.18) is feasible for $(n, (F_i))_{i=1}^n$, and $\min_{i \in [n]} \frac{c_f(i) + c_b(i)}{2} = \frac{1}{3}$.

Theorem 1.4 now follows easily.

Proof of Theorem 1.4. Lemma 4.4 implies that $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ as defined in (4.18) is feasible for $(n, (F_{i})_{i=1}^{n})$. Thus, by using them in Algorithm 3, Theorem 4.1 implies that for each $\sigma \in \{\mathsf{f}, \mathsf{b}\}$, $i \in [n]$ and $s_{i} \in [0, 1]$, $\Pr[A_{i} = 1 \mid \Lambda = \sigma, S_{i} = s_{i}] = c_{\sigma}(i)$. As such, since Λ is uniformly distributed on $\{\mathsf{f}, \mathsf{b}\}$, Algorithm 3 is $\min_{i \in [n]} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2$ -selectable. Since Lemma 4.4 ensures $\min_{i \in [n]} \frac{c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i)}{2} = \frac{1}{3}$, the proof is complete.

4.2 Knapsack FB-CRS Hardness Result: Proving Theorem 1.5

In order to prove Theorem 1.5, for each $n \in \mathbb{N}$, we set $\rho = 2n/(n+2)$, and consider the knapsack input $(2n + 1, (F_i)_{i=1}^{2n+1})$, where $S_i \in \{1/2 + 1/n, \infty\}$ and $\Pr[S_i = 1/2 + 1/n] = \rho/(2n+1)$ for each $i \in [2n+1]$. Instead of directly trying to analyze the performance of a knapsack FB-CRS on $(2n + 1, (F_i)_{i=1}^{2n+1})$, we consider the single-unit input $(2n + 1, \mathbf{x})$, where $x_i = \rho/(2n+1)$ for each $i \in [2n+1]$, and make the following observation.

Proposition 4.5. There exists an α -selectable knapsack FB-CRS for $(2n+1, (F_i)_{i=1}^{2n+1})$ if and only if there exists an α -selectable single-unit FB-CRS for $(2n+1, \mathbf{x})$.

Proof of Proposition 4.5. Since the support of each F_i is $\{1/2 + 1/n, \infty\}$, and 1/2 + 1/n > 1/2, at most one element can be accepted by any knapsack FB-CRS. The claim thus follows immediately.

Using this observation, we can now apply our negative results from Section 3, namely Lemma 3.1 and Theorem 3.4, to derive Theorem 1.5.

Proof of Theorem 1.5. It suffices to upper bound the selection guarantee of an arbitrary single-unit FB-CRS on $(2n + 1, \mathbf{x})$. Now, by applying Lemma 3.1 to $(2n + 1, \mathbf{x})$, we know that no single-unit FB-CRS can attain a selection guarantee greater than LPOPT $(2n + 1, \mathbf{x})$. However, $(2n + 1, \mathbf{x})$ is precisely the input described in Theorem 3.4 for the parameter ρ , and so we know that

LPOPT
$$(2n+1, \mathbf{x}) \le \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho} + \frac{\rho+2}{2n+1} = (1+o(1))\frac{1}{2+e^{-1}}$$

where the asymptotics hold since $\rho \to 2$ as $n \to \infty$. The proof is thus complete.

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A Additions to Section 2

A.1 Proof of Lemma 2.1

Let $\beta_i = \mathbb{E}[s_i(Y_i, D_i)]$ for all *i*. We set q_i to the smallest value in [0,1] that makes (2.2) hold. To see that such a value exists, note that if $q_i = 1$, then the left-hand side (LHS) of (2.2) equals $\mathbb{E}[s_i(\min\{D_i, 1\}, D_i)]$, which is an upper bound on $\mathbb{E}[s_i(Y_i, D_i)] = \beta_i$ as $Y_i \leq \min\{D_i, 1\}$. The LHS of (2.2) continuously decreases to 0 as q_i decreases from 1 to 0, and hence this value of q_i exists. Under this definition of q_i , we now prove that

$$\mathbb{E}[Y_i] \ge \int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq,$$
(A.1)

which will help us establish (2.1).

If s_i is the Type-I service function, then (2.2) implies $\beta_i \leq \min\{q_i, F_i(1)\}$ (see (2.3)). However in this case we know $q_i \leq F_i(1)$ by virtue of q_i being the smallest value that satisfies (2.2) (increasing q_i beyond $F_i(1)$ does increase the LHS of (2.2), under Type-I service). We derive

$$\mathbb{E}[Y_i] \ge \mathbb{E}[\mathbb{1}(Y_i = D_i)D_i] \ge \int_0^{\Pr[Y_i = D_i]} F_i^{-1}(q) dq$$

by the optimality of monotone coupling between $\mathbb{1}(Y_i = D_i)$ and demand D_i being small. Using the facts that $\Pr[Y_i = D_i] = \beta_i = q_i$ and that $F_i^{-1}(q) \leq 1$ for all q below this q_i , the right-hand side (RHS) of the preceding equation equals $\int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq$, establishing (A.1).

If s_i is the Type-II service function, then (2.2) implies $\int_0^{q_i} (\min\{F_i^{-1}(q), 1\}/\mu_i) dq = \beta_i = \mathbb{E}[Y_i]/\mu_i$, which immediately establishes (A.1) as equality.

If s_i is the Type-III service function, then we have

$$\beta_i = \mathbb{E}[\frac{Y_i}{D_i}] = \int_0^1 \frac{\mathbb{E}[Y_i \mid Q_i = q]}{F_i^{-1}(q)} dq \le \int_0^{q'_i} \frac{\min\{F_i^{-1}(q), 1\}}{F_i^{-1}(q)} dq$$

where q'_i is such that $\int_0^{q'_i} \min\{F_i^{-1}(q), 1\} dq = \mathbb{E}[Y_i]$. This again follows by optimality of monotone coupling between the distributions Y_i and D_i : the coefficient $1/F_i^{-1}(q)$ is maximized when q is small, and hence we also want $\mathbb{E}[Y_i \mid Q_i = q]$ to be maximized when q is small, subject to $\mathbb{E}[Y_i \mid Q_i = q] \leq F_i^{-1}(q)$ and $\mathbb{E}[Y_i \mid Q_i = q] \leq 1$. Now, because $\beta_i = \int_0^{q_i} \frac{\min\{F_i^{-1}(q), 1\}}{F_i^{-1}(q)} dq$, we deduce that $q'_i \geq q_i$. But then $\mathbb{E}[Y_i] \geq \int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq$, establishing (A.1).

Having established (A.1) for all three types of service, we now use the fact that $\sum_{i=1}^{n} Y_i \leq 1$ on every sample path, and take linearity of expectation. We get

$$1 \ge \sum_{i=1}^{n} \mathbb{E}[Y_i] \ge \sum_{i=1}^{n} \int_0^{q_i} \min\{F_i^{-1}(q), 1\} dq,$$

which establishes (2.1) as desired.

B Additions to Section 3

B.1 Proof of Lemma 3.1

Recall that A_i is an indicator random variable for the event that $i \in [n]$ is accepted by Algorithm 2. Our goal is to show that for each $i \in [n]$ and $\sigma \in \{f, b\}$,

$$\Pr[A_i = 1 \mid S_i = 1, \Lambda = \sigma] = c_{\sigma}(i). \tag{B.1}$$

We first prove (B.1) for $\sigma = f$ using induction on the elements of [n]. Observe first that for i = 1, if we condition on $\Lambda = f$ and $S_1 = 1$, then element 1 is accepted if and only if $B_f(1) = 1$. Thus,

$$\Pr[A_1 = 1 \mid S_1 = 1, \Lambda = \mathsf{f}] = \Pr[B_\mathsf{f}(1) = 1 \mid \Lambda = \mathsf{f}] = c_\mathsf{f}(1).$$

For $\sigma = f$ and i > 1, let us now assume that (B.1) holds for all j < i. Observe then that since at most one element is accepted by Algorithm 2,

$$\Pr[\bigcup_{j < i} \{A_j = 1\} \mid \Lambda = \mathsf{f}] = \sum_{j < i} c_{\mathsf{f}}(j) \cdot x_j.$$
(B.2)

On the other hand, conditional on $S_i = 1$ and $\Lambda = f$, *i* is accepted if and only if $\bigcap_{j < i} \{A_j = 0\}$ and $B_f(i) = 1$. Since conditional on $\Lambda = f$, $B_f(i)$ is independent of $(A_j)_{j < i}$ and S_i , we have that

$$\Pr[A_i = 1 \mid S_i = 1, \Lambda = \mathsf{f}] = \Pr[B_\mathsf{f}(i) = 1 \mid \Lambda = \mathsf{f}] \cdot (1 - \Pr[\cup_{j < i} \{A_j = 1\} \mid \Lambda = \mathsf{f}])$$
$$= \frac{c_\mathsf{f}(i)}{1 - \sum_{j < i} x_j \cdot c_\mathsf{f}(j)} \cdot \left(1 - \sum_{j < i} x_j \cdot c_\mathsf{f}(j)\right)$$
$$= c_\mathsf{f}(i),$$

where the second equality follows from (B.2). Thus, (B.1) holds for $\sigma = f$ for all $i \in [n]$. We omit the case when $\sigma = b$, as the argument proceeds identically. Using (B.1), and the fact that $\Pr[\Lambda = f] = \Pr[\Lambda = b] = 1/2$, we have that for each $i \in [n]$,

$$\Pr[A_i = 1 \mid S_i = 1] = (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2.$$

The selection guarantee of Algorithm 2 is therefore $\min_{1 \le i \le n} (c_{f}(i) + c_{b}(i))/2 = \text{LPOPT}(n, \boldsymbol{x}).$

Suppose now that we have an arbitrary FB-CRS for (n, \mathbf{x}) . Let us first set $c_{\sigma}(i) := \Pr[A_i = 1 | S_i = 1, \Lambda = \sigma]$ for each $i \in [n]$ and $\sigma \in \{f, b\}$, where A_i is the indicator random variable for the event the FB-CRS accepts i. Observe then that since Λ is distributed uniformly on $\{f, b\}$,

$$\Pr[A_i = 1 \mid S_i = 1] = (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2,$$

Thus, the selection guarantee of the FB-CRS is $\min_{1 \le i \le n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2$. To complete the proof, it suffices to show that $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ is a feasible solution to LP-SI.

Clearly, $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ is non-negative. Now, fix $i \in [n]$ and condition on $\Lambda = \sigma$ and $S_i = 1$. Observe that if *i* is accepted, then no $j \in [n]$ with $\sigma(j) < \sigma(i)$ could have previously been accepted. Thus,

$$\begin{aligned} c_{\sigma}(i) &= \Pr[A_i = 1 \mid S_i = 1, \Lambda = \sigma] \leq 1 - \Pr[\cup_{j < \sigma^i} \{A_j = 1\} \mid \Lambda = \sigma, S_i = 1] \\ &= 1 - \sum_{j < \sigma^i} \Pr[A_j = 1 \mid \Lambda = \sigma, S_i = 1] \\ &= 1 - \sum_{j < \sigma^i} \Pr[A_j = 1 \mid \Lambda = \sigma] \\ &= 1 - \sum_{j < \sigma^i} c_{\sigma}(j) \cdot x_j. \end{aligned}$$

Here the first equality uses that at most one element can be accepted, the second uses that S_i is independent of A_j (conditional on $\Lambda = \sigma$), and the final uses the definition of $c_{\sigma}(j)$. Thus, (3.1) holds, and so the proof is complete.

B.2 Proof of Proposition 3.2

Denote $\alpha_0 = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho}$, and recall that $\phi : [0, \rho] \to [0, 1]$, where for $z \in [0, \rho]$,

$$\phi(z) := \begin{cases} \frac{2e^{\rho/2} - e^z}{1 + e^{\rho/2}\rho} & \text{if } z \le \rho/2, \\ \frac{e^{\rho-z}}{1 + e^{\rho/2}\rho} & \text{if } \rho/2 < z \le \rho. \end{cases}$$
(B.3)

We verify the properties of Proposition 3.2 in order. Since $\lim_{z\to(\rho/2)^-} \phi(z) = \lim_{z\to(\rho/2)^+} \phi(z) = \alpha_0$, it is clear that ϕ is continuous. More, its derivative on $[0, \rho/2)$ and $(\rho/2, \rho]$ is negative, so it is decreasing.

Now, we already know $(\phi(z) + \phi(\rho - z))/2 = \alpha_0$ for $z = \rho/2$. Observe that for any $z \in [0, \rho/2)$, $\rho/2 < \rho - z \le \rho$, and so

$$\frac{\phi(z) + \phi(\rho - z)}{2} = \frac{1}{2} \left(\frac{2e^{\rho/2} - e^z}{1 + e^{\rho/2}\rho} + \frac{e^{\rho - (\rho - z)}}{1 + e^{\rho/2}\rho} \right) = \frac{\exp(\rho/2)}{1 + \exp(\rho/2)\rho} = \alpha_0$$

The same applies for $z > \rho/2$, due to the symmetry of (B.3). Next, observe that for $z \le \rho/2$,

$$\phi(z) + \int_0^z \phi(\tau) d\tau = \frac{2e^{\rho/2} - e^z}{1 + e^{\rho/2}\rho} + \frac{1 - e^z + 2e^{\rho/2}z}{1 + e^{\rho/2}\rho} = \frac{1 - 2e^z + 2e^{\rho/2}(1+z)}{1 + e^{\rho/2}\rho} \le 1,$$

where the inequality follows since the maximum of the function of z on the LHS occurs at $z = \rho/2$ (where it in fact takes a value of 1). Finally, for $z > \rho/2$,

$$\phi(z) + \int_0^{\rho/2} \phi(\tau) d\tau + \int_{\rho/2}^z \phi(\tau) d\tau = \frac{e^{\rho-z}}{1 + e^{\rho/2}\rho} + \frac{e^{\rho/2} - e^{\rho-z}}{1 + e^{\rho/2}\rho} + \frac{1 + e^{\rho/2}(\rho-1)}{1 + e^{\rho/2}\rho} = 1$$

Thus, the proof is complete.

B.3 Single-unit LP Duality Details

We first state LP-SI as a linear program in standard form by introducing an additional variable β , and rearranging the inequalities:

maximize
$$\beta$$
 (LP-SI-A)
subject to $\beta - (c_{\mathbf{f}}(i) + c_{\mathbf{b}}(i))/2 \leq 0$ $\forall i \in [n]$
 $c_{\sigma}(i) + \sum_{j < \sigma i} x_j \cdot c_{\sigma}(j) \leq 1$ $\forall i \in [n], \sigma \in \{\mathbf{f}, \mathbf{b}\}$
 $c_{\sigma}(i) \geq 0$ $\forall i \in [n], \sigma \in \{\mathbf{f}, \mathbf{b}\}$

After taking its dual, we get the following:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} (y_{\mathsf{f}}(i) + y_{\mathsf{b}}(i)) \\ \text{subject to} & y_{\sigma}(i) + \sum_{j > \sigma^{i}} \frac{\rho \cdot y_{\sigma}(j)}{N} - \frac{\xi(i)}{2} \geq 0 & \forall i \in [N], \sigma \in \{\mathsf{f}, \mathsf{b}\} \\ & \sum_{i=1}^{N} \xi(i) \geq 1 \end{array}$$

$$\xi(i), y_{\mathsf{f}}(i), y_{\mathsf{b}}(i) \ge 0 \qquad \qquad \forall i \in [n].$$

By scaling a solution to this dual up by N, we can write it in the following way, as presented in dual-LP-SI of Subsection 3.2:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \frac{(y_{\mathsf{f}}(i) + y_{\mathsf{b}}(i))}{N} \\ \text{subject to} & y_{\sigma}(i) + \sum_{j > \sigma^{i}} \frac{\rho \cdot y_{\sigma}(j)}{N} - \frac{\xi(i)}{2} \ge 0 & \forall i \in [N], \sigma \in \{\mathsf{f}, \mathsf{b}\} \\ & \sum_{i=1}^{n} \frac{\xi(i)}{N} \ge 1 \\ & \xi(i), y_{\mathsf{f}}(i), y_{\mathsf{b}}(i) \ge 0 & \forall i \in [N]. \end{array}$$

Proof of Proposition 3.6 B.4

 \mathbf{S}

Recall that $\alpha_0 := \frac{\exp(\rho/2)}{1+\exp(\rho/2)\rho}$, and $\gamma(z) := \frac{\rho \exp(z-\rho/2)}{2(1+\exp(\rho/2)\rho)}$. Observe that $\gamma'(z) = \gamma(z)$ for $z \in [\rho/2,\rho]$, so clearly γ is increasing on $[\rho/2,\rho]$. More, it is 1-Lipschitz, since $\max_{z \in [\rho/2,\rho]} |\gamma'(z)| = 1$ $\max_{z \in [\rho/2,\rho]} |\gamma(z)| = \gamma(\rho) = \frac{p\alpha_0}{2} \le 1$. Finally,

$$\gamma(z) + \int_{z}^{\rho} \gamma(\tau) d\tau = \frac{\rho \alpha_{0}}{2},$$

is easily verified for $z \in [\rho/2, \rho]$, as $\gamma(z)$ is the unique solution to the differential equation $\gamma'(z) =$ $\gamma(z)$, with initial condition $\gamma(\rho) = \frac{\rho \alpha_0}{2}$.

Splitting Argument B.5

Suppose that (n, \mathbf{x}) is an arbitrary input with $\sum_{i=1}^{n} x_i = \rho$. Now, if we take any $\varepsilon > 0$, then we claim that there exists an input (\tilde{n}, \tilde{x}) with $\tilde{x}_i \leq \varepsilon$ for all $i \in [\tilde{n}], \sum_{i=1}^{\tilde{n}} \tilde{x}_i = \rho$, and

$$\text{LPOPT}(\tilde{n}, \tilde{x}) \leq \text{LPOPT}(n, x).$$

In order to prove this, it suffices to prove the following *splitting argument*. The claim then follows by applying this lemma a finite number of times.

Lemma B.1 (Splitting Argument). Given an input (n, \mathbf{x}) and an index $k \in [n]$, construct an input $(n+1, \tilde{x})$, where $\tilde{x}_i := x_i$ for $1 \le i < k$, $\tilde{x}_k = \tilde{x}_{k+1} := x_k/2$, and $\tilde{x}_i := x_{i-1}$ for $k+1 < i \le n+1$. Then, $\sum_{i=1}^{n+1} \tilde{x}_i = \sum_{i=1}^n x_i$, and $LPOPT(n+1, \tilde{x}) \le LPOPT(n, x)$.

Proof. Given $(n+1, \tilde{x})$, suppose that $(\tilde{c}_{\mathsf{f}}(i), \tilde{c}_{\mathsf{b}}(i))_{i=1}^{n+1}$ is an optimal solution to LP-SI. For input (n, \boldsymbol{x}) , we shall construct a feasible solution $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^{n}$ to LP-SI such that

$$\min_{1 \le i \le n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i)) \ge \min_{1 \le i \le n+1} (\tilde{c}_{\mathsf{f}}(i) + \tilde{c}_{\mathsf{b}}(i)).$$
(B.4)

This will complete the proof, as the feasibility implies that $LPOPT(n, x) \geq \min_{1 \leq i \leq n} (c_f(i) + i)$ $c_{\mathsf{b}}(i))/2$, and so combined with (B.4),

$$\mathrm{LPOPT}(n, \boldsymbol{x}) \ge \min_{1 \le i \le n} (c_{\mathsf{f}}(i) + c_{\mathsf{b}}(i))/2 \ge \min_{1 \le i \le n+1} (\tilde{c}_{\mathsf{f}}(i) + \tilde{c}_{\mathsf{b}}(i))/2 = \mathrm{LPOPT}(n+1, \boldsymbol{\tilde{x}}).$$

For each $\sigma \in \{f, b\}$, we define $c_{\sigma}(i)$ based on the following cases:

$$c_{\sigma}(i) := \begin{cases} \tilde{c}_{\sigma}(i) & \text{if } i < k, \\ (\tilde{c}_{\sigma}(k) + \tilde{c}_{\sigma}(k+1))/2 & \text{if } i = k, \\ \tilde{c}_{\sigma}(i+1) & \text{if } k < i \le n. \end{cases}$$
(B.5)

First observe that

$$c_{f}(i) + c_{b}(i) = \begin{cases} \tilde{c}_{f}(i) + \tilde{c}_{b}(i) & \text{if } i < k, \\ \tilde{c}_{f}(i+1) + \tilde{c}_{b}(i+1) & \text{if } k < i \le n. \end{cases}$$
(B.6)

On the other hand,

$$c_{f}(k) + c_{b}(k) = (\tilde{c}_{f}(k) + \tilde{c}_{f}(k+1))/2 + (\tilde{c}_{b}(k) + \tilde{c}_{b}(k+1))/2$$

= $(\tilde{c}_{f}(k) + \tilde{c}_{b}(k+1))/2 + (\tilde{c}_{f}(k) + \tilde{c}_{b}(k+1))/2$
 $\geq \min\{\tilde{c}_{f}(k) + \tilde{c}_{b}(k+1), \tilde{c}_{f}(k) + \tilde{c}_{b}(k+1)\}.$ (B.7)

Thus, (B.6) and (B.7) immediately imply (B.4).

We shall now argue that $(c_{f}(i), c_{b}(i))_{i=1}^{n}$ is a feasible solution to LP-SI. We focus on verifying (3.1) of LP-SI for $\sigma = f$, as the case of $\sigma = b$ proceeds identically. First observe that since $(\tilde{c}_{f}(i), \tilde{c}_{b}(i))_{i=1}^{n+1}$ is a feasible solution to LP-SI, we have that for all $i \in [n+1]$,

$$\tilde{c}_{\mathsf{f}}(i) + \sum_{j < i} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j \le 1 \tag{B.8}$$

Now, for i < k, $c_{\mathsf{f}}(i) + \sum_{j < i} c_{\mathsf{f}}(j) x_j = \tilde{c}_{\mathsf{f}}(i) + \sum_{j < i} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j$, so (3.1) of LP-SI immediately holds due to (B.8). For i = k, we first observe that since $c_{\mathsf{f}}(k) = (\tilde{c}_{\mathsf{f}}(k) + \tilde{c}_{\mathsf{f}}(k+1))/2$,

$$c_{\mathsf{f}}(k) \le \max\{\tilde{c}_{\mathsf{f}}(k), \tilde{c}_{\mathsf{f}}(k+1)\},\$$

and so $c_{\rm f}(k) \leq \tilde{c}_{\rm f}(k)$ or $c_{\rm f}(k) \leq \tilde{c}_{\rm f}(k+1)$. We handle both cases separately. If $c_{\rm f}(k) \leq \tilde{c}_{\rm f}(k)$, then

$$c_{\mathsf{f}}(k) + \sum_{j < k} c_{\mathsf{f}}(j) x_j = c_{\mathsf{f}}(k) + \sum_{j < k} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j \le \tilde{c}_{\mathsf{f}}(k) + \sum_{j < k} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j \le 1,$$

where the final inequality applies (B.8) with i = k. On the other hand, if $c_{\rm f}(k) \leq \tilde{c}_{\rm f}(k+1)$, then

$$c_{\mathsf{f}}(k) + \sum_{j < k} c_{\mathsf{f}}(j) x_j = c_{\mathsf{f}}(k) + \sum_{j < k} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j \le \tilde{c}_{\mathsf{f}}(k+1) + \sum_{j \le k} \tilde{c}_{\mathsf{f}}(j) \tilde{x}_j \le 1$$

where the final inequality applies (B.8) with i = k + 1. It remains to verify the case when i > k. Observe that since $c_{\rm f}(k) = (\tilde{c}_{\rm f}(k) + \tilde{c}_{\rm f}(k+1))/2$ and $\tilde{x}_k = \tilde{x}_{k+1} = x_k/2$,

$$c_{\mathsf{f}}(k)x_k = \tilde{c}_{\mathsf{f}}(k)\tilde{x}_k + \tilde{c}_{\mathsf{f}}(k+1)\tilde{x}_{k+1}.$$
(B.9)

Thus, using the definition of $(c_f(j))_{j \leq i}$,

$$\begin{split} c_{f}(i) + \sum_{j < i} c_{f}(j) x_{j} &= \tilde{c}_{f}(i+1) + \sum_{j < k} \tilde{c}_{f}(j) \tilde{x}_{j} + c_{f}(k) x_{k} + \sum_{k+1 < j \le i} \tilde{c}_{f}(j) \tilde{x}_{j} \\ &= \tilde{c}_{f}(i+1) + \sum_{j < k} \tilde{c}_{f}(j) \tilde{x}_{j} + \tilde{c}_{f}(k) \tilde{x}_{k} + \tilde{c}_{f}(k+1) \tilde{x}_{k+1} + \sum_{k+1 < j \le i} \tilde{c}_{f}(j) \tilde{x}_{j} \\ &= \tilde{c}_{f}(i+1) + \sum_{j \le i} \tilde{c}_{f}(j) \tilde{x}_{j} \le 1, \end{split}$$

where the second equality follows by (B.9), and the final inequality applies (B.8). Thus, $(c_{f}(i), c_{b}(i))_{i=1}^{n}$ is a feasible solution to LP-SI, and so the proof is complete.

C Additions to Section 4

C.1 Induction Proof

We proceed inductively, beginning with the base case for element $i_1 = \sigma^{-1}(1) \in \{1, n\}$:

Lemma C.1 (Base Case). Fix $\sigma \in \{f, b\}$. Then, (4.6) and (4.7) hold for $i_1 = \sigma^{-1}(1)$.

Proof of Lemma C.1. For $\sigma \in \{f, b\}$, we verify (4.6) and (4.7) for $i_1 = \sigma^{-1}(1)$. Observe that for any $0 < b \leq 1/2$, since $\Pr[0 < \mathcal{T}_{\sigma}(i_1) \leq b \mid \Lambda = \sigma] = 0$, the LHS of (4.6) is 0, and so (4.6) holds. Similarly, since $\Pr[\mathcal{T}_{\sigma}(i_1) = 0 \mid \Lambda = \sigma] = 1$ and $c_{\sigma}(i_1) \leq 1$ by Definition 5, (4.7) holds.

We now complete the inductive step for (4.6). This is similar to the proof of Lemma 4 in [JMZ22].

Lemma C.2 (Inductive Step for (4.6)). Fix $\sigma \in \{f, b\}$ and $i \in [n]$ with $i_1 <_{\sigma} i$. If (4.6) and (4.7) hold for all $j <_{\sigma} i$, then (4.6) holds for i.

Proof of Lemma C.2. Let us condition on $\Lambda = f$. We prove the claim for this case, as when $\Lambda = b$, the argument proceeds identically. In order to simplify the notation, we implicitly condition on $\Lambda = f$ in all of our computations. More, to be consistent with the indexing in Algorithm 3, we assume that (4.6) and (4.7) hold for all $1 \leq j \leq i$, and prove that (4.6) holds for i + 1. Fix $0 < b \leq 1/2$, and observe that due to the induction hypothesis, we know that

$$\frac{\Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b]}{c_{\mathsf{f}}(1)} \le \exp\left(-\frac{\Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b]}{c_{\mathsf{f}}(1)}\right). \tag{C.1}$$

In order to extend this to i + 1, we have to consider what happens when Algorithm 3 processes element *i*, as this will determine how the *distribution* of $\mathcal{T}_{f}(i+1)$ differs from the distribution of $\mathcal{T}_{f}(i)$. We refer to $s_i \in [0, 1]$ as 0-avoiding, provided $c_f(i) \leq \Pr[0 < \mathcal{T}_f(i) \leq 1 - s_i]$. Otherwise, if $c_f(i) > \Pr[0 < \mathcal{T}_f(i) \leq 1 - s_i]$, then we refer to s_i as 0-using. Now, recalling the definition of $B_f(i)$ from Algorithm 3, we know that if s_i is 0-avoiding, then

$$\Pr[B_{f}(i) = 1 \mid 0 < \mathcal{T}_{f}(i) \le 1 - s_{i}] = \frac{c_{f}(i)}{\Pr[0 < \mathcal{T}_{f}(i) \le 1 - s_{i}]}$$
$$\Pr[B_{f}(i) = 1 \mid \mathcal{T}_{f}(i) = 0] = 0$$

Otherwise, if s_i is 0-using, then

$$\Pr[B_{f}(i) = 1 \mid 0 < \mathcal{T}_{f}(i) \le 1 - s_{i}] = 1.$$

$$\Pr[B_{f}(i) = 1 \mid \mathcal{T}_{f}(i) = 0] = \frac{c_{f}(i) - \Pr[0 < \mathcal{T}_{f}(i) \le 1 - s_{i}]}{\Pr[\mathcal{T}_{f}(i) = 0]}$$

(Observe that the final fraction is at most 1, since we assumed (4.7) holds for i, and so (4.8) of Lemma 4.2 applies). Thus, when s_i is 0-avoiding, we never accept i when $\mathcal{T}_{f}(i) = 0$. Conversely, when i is 0-using, there is a non-zero probability that i is accepted when $\mathcal{T}_{f}(i) = 0$. We further classify $s_i \in [0, 1]$:

$$S_{i,1} = \{s_i \in (0, b] : s_i \text{ is } 0\text{-using}\}$$

$$S_{i,2} = \{s_i \in (b, 1 - b] : s_i \text{ is } 0\text{-using}\}$$

$$S_{i,3} = \{s_i \in (0, 1 - b] : s_i \text{ is } 0\text{-avoiding}\}$$

$$\mathcal{S}_{i,4} = (1-b,1] \cup \{0\}$$

Before continuing, we define two functions on [0, 1], whose usage will become clear below:

$$a_1(s) := \begin{cases} \Pr[b - s < \mathcal{T}_{\mathsf{f}}(i) \le b] & \text{if } s \in \mathcal{S}_{i,1}.\\ 0 & \text{if } s \in [0,1] \setminus \mathcal{S}_{i,1}. \end{cases}$$
(C.2)

$$a_{3}(s) := \begin{cases} \frac{\Pr[b-s < \mathcal{T}_{\mathbf{f}}(i) \le b] \cdot c_{\mathbf{f}}(i)}{\Pr[0 < \mathcal{T}_{\mathbf{f}}(i) \le 1-s]} & \text{if } s \in \mathcal{S}_{i,3}.\\ 0 & \text{if } s \in [0,1] \setminus \mathcal{S}_{i,3}. \end{cases}$$
(C.3)

(Here (C.3) is well-defined, since $c_{\mathsf{f}}(i) \leq \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \leq 1-s]$ for $s \in \mathcal{S}_{i,3}$). We now upper bound $\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \leq b \mid S_i = s_i]$ and $\Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \leq 1-b \mid S_i = s_i]$ for the various classifications of s_i . In the explanations, we implicitly condition on $S_i = s_i$, which we note is independent of $\mathcal{T}_{\mathsf{f}}(i)$.

If $s_i \in S_{i,1}$: Observe $0 < \mathcal{T}_{f}(i+1) \leq b$ occurs if and only if $0 < \mathcal{T}_{f}(i) \leq b - s_i$, or $\{B_f(i) = 1\} \cap \{\mathcal{T}_f(i) = 0\}$. Now, due to the definition of $B_f(i)$ when s_i is 0-using, this final event occurs with probability

$$c_{\mathsf{f}}(i) - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le 1 - s_i] \le c_{\mathsf{f}}(i) - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b],$$

where the inequality uses $1-b \leq 1-s_i$. Similarly, if $b < \mathcal{T}_{f}(i+1) \leq 1-b$ occurs, then $b < \mathcal{T}_{f}(i) \leq 1-b$ or $b-s_i < \mathcal{T}_{f}(i) \leq b$. Thus, using the definition of a_1 from (C.2):

$$\begin{aligned} &\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b \mid S_i = s_i] \le \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b] - a_1(s_i) + (c_{\mathsf{f}}(i) - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b]), \\ &\Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1 - b \mid S_i = s_i] \le \Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b] + a_1(s). \end{aligned}$$

If $s_i \in S_{i,2}$: Since $b < s_i \le 1-b$ and s_i is 0-using, we know that $0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b$ cannot occur. On the other hand, $b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1-b$ occurs only if $0 < \mathcal{T}_{\mathsf{f}}(i) \le 1-b$, or $\{\mathcal{T}_{\mathsf{f}}(i) = 0\} \cap \{B_{\mathsf{f}}(i) = 1\}$. Due to the definition of $B_{\mathsf{f}}(i)$ when s_i is 0-using, this final event occurs with probability

$$c_{\mathsf{f}}(i) - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le 1 - s_i] \le c_{\mathsf{f}}(i) - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b],$$

where the inequality holds since $1 - s_i \ge b$. Thus,

$$\begin{aligned} &\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b \mid S_i = s_i] = 0, \\ &\Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1 - b \mid S_i = s_i] \le \Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b] + c_{\mathsf{f}}(i). \end{aligned}$$

If $s_i \in S_{i,3}$: Observe $0 < \mathcal{T}_{\mathsf{f}}(i+1) \leq b$ occurs if and only if $0 < \mathcal{T}_{\mathsf{f}}(i) \leq b - s_i$, or $\{b - s_i < \mathcal{T}_{\mathsf{f}}(i) \leq b\} \cap \{B_{\mathsf{f}}(i) = 0\}$. On the other hand, since s_i is 0-avoiding, the probability of the latter event is

$$\Pr[b - s_i < \mathcal{T}_{\mathsf{f}}(i) \le b] \cdot \left(1 - \frac{c_{\mathsf{f}}(i)}{\Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le 1 - s_i]}\right)$$

More, $b < \mathcal{T}_{f}(i+1) \leq 1-b$ occurs only if $b < \mathcal{T}_{f}(i) \leq 1-b$ or $\{b-s_i < \mathcal{T}_{f}(i) \leq b\} \cap \{B_{f}(i)=1\}$. Thus, recalling the definition of a_3 from (C.3), we have that

$$\Pr[0 < \mathcal{T}_{f}(i+1) \le b \mid S_{i} = s_{i}] \le \Pr[0 < \mathcal{T}_{f}(i) \le b] - a_{3}(s_{i}),$$

$$\Pr[b < \mathcal{T}_{f}(i+1) \le 1 - b \mid S_{i} = s_{i}] \le \Pr[b < \mathcal{T}_{f}(i) \le 1 - b] + a_{3}(s_{i}).$$

If $s_i \in \mathcal{S}_{i,4}$: Then, since $s_i = 0$ or $s_i > 1 - b$, the relevant probabilities are non-increasing.

$$\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b \mid S_i = s_i] \le \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b],$$

$$\Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1 - b \mid S_i = s_i] \le \Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1 - b].$$

We now define $\tilde{a}_1 := \mathbb{E}[a_1(S_i)]$, $\tilde{a}_3 := \mathbb{E}[a_3(S_i)]$, and $p_k := \Pr[S_i \in S_{i,k}]$ for $k \in [4]$. Using the upper bounds for $\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \leq b \mid S_i = s_i]$ and averaging over S_i , after simplification we get that:

$$\Pr[0 < \mathcal{T}_{f}(i+1) \le b] \le \Pr[0 < \mathcal{T}_{f}(i) \le b] + (c_{f}(i) - \Pr[0 < \mathcal{T}_{f}(i) \le 1 - b])p_{1} - \tilde{a}_{1} - \Pr[0 < \mathcal{T}_{f}(i) \le b]p_{2} - \tilde{a}_{3}$$
(C.4)

Similarly, after applying the upper bounds for $\Pr[b < \mathcal{T}_{f}(i+1) \le 1-b \mid S_i = s_i]$ and averaging over S_i :

$$\Pr[b < \mathcal{T}_{f}(i+1) \le 1-b] \le \Pr[b < \mathcal{T}_{f}(i) \le 1-b] + \tilde{a}_{1} + c_{f}(i)p_{2} + \tilde{a}_{3}.$$
 (C.5)

The remaining computations are mostly algebraic and follow the derivation of Lemma 4 in [JMZ22], however we sketch the main steps for completeness. Let us first consider when $p_1 = 0$. In this case, $\tilde{a}_1 = 0$, and so applied to (C.4), we get that

$$\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b] \le \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b] - \Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b]p_2 - \tilde{a}_3.$$
(C.6)

Moreover, using $\tilde{a}_1 = 0$ and $c_f(i) \le c_f(1)$ (due to Definition 5), (C.5) simplifies to

 \geq

$$\Pr[b < \mathcal{T}_{f}(i+1) \le 1-b] \le \Pr[b < \mathcal{T}_{f}(i) \le 1-b] + c_{f}(1)p_{2} + \tilde{a}_{3}$$
(C.7)

Thus, applying the elementary bounds of $1 - z \le \exp(-z)$ and $\exp(-z) \le 1$ to (C.7), followed by (C.1) (our induction hypothesis)

$$\exp\left(-\frac{\Pr[b < \mathcal{T}_{f}(i+1) \le 1-b]}{c_{f}(1)}\right) \ge \exp\left(-\frac{\Pr[b < \mathcal{T}_{f}(i) \le 1-b]}{c_{f}(1)}\right)(1-p_{2}) - \frac{\tilde{a}_{3}}{c_{f}(1)}$$
(C.8)

$$\frac{\Pr[0 < \mathcal{T}_{\mathsf{f}}(i) \le b]}{c_{\mathsf{f}}(1)} (1 - p_2) - \frac{\tilde{a}_3}{c_{\mathsf{f}}(1)}.$$
(C.9)

By combining this with (C.6) (after dividing by $c_{f}(1)$), we have extended (C.1) to i + 1 as desired. It remains to consider when $p_1 > 0$. Then, since $p_1 \le 1 - p_2$ and $c_{f}(i) \le c_{f}(1)$, we can write (C.4) as

$$\Pr[0 < \mathcal{T}_{\mathsf{f}}(i+1) \le b] \le (c_{\mathsf{f}}(1) - \Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1 - b])(1 - p_2) - \tilde{a}_1 - \tilde{a}_3.$$
(C.10)

Moreover, by using $c_{f}(i) \leq c_{f}(1)$ and the same elementary bounds as before,

$$\exp\left(-\frac{\Pr[b < \mathcal{T}_{\mathsf{f}}(i+1) \le 1-b]}{c_{\mathsf{f}}(i)}\right) \ge \exp\left(-\frac{\Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1-b]}{c_{\mathsf{f}}(i)} - p_2\right) - \frac{\tilde{a}_1}{c_{\mathsf{f}}(1)} - \frac{\tilde{a}_3}{c_{\mathsf{f}}(1)} \\ \ge \left(1 - \frac{\Pr[b < \mathcal{T}_{\mathsf{f}}(i) \le 1-b]}{c_{\mathsf{f}}(1)}\right)(1-p_2) - \frac{\tilde{a}_1}{c_{\mathsf{f}}(1)} - \frac{\tilde{a}_3}{c_{\mathsf{f}}(1)}.$$
(C.11)

Thus, after dividing (C.10) by $c_f(1)$ and applying (C.11), we have extended (C.1) to i + 1 as desired.

The next lemma provides the details of how to use (4.6) to simplify the integral in (4.13). This is essentially the proof of Theorem 5 from [JMZ22]; however we include it for completeness.

Lemma C.3 (Inductive Step for (4.7)). Fix $\sigma \in \{f, b\}$ and $i \in [n]$ with $i_1 <_{\sigma} i$. If (4.6) and (4.7) hold for all $j <_{\sigma} i$, and (4.6) holds for i, then (4.7) holds for i.

Remark 9. We can assume that (4.6) holds for *i*, due to Lemma C.2.

Proof of Lemma C.3. Let us condition on $\Lambda = f$. We prove the claim for this case, as when $\Lambda = b$, the argument proceeds identically. For each $0 < \tau \leq 1$, define $U_i(\tau) := \Pr[0 < \mathcal{T}_{f}(i) \leq \tau \mid \Lambda = f]$. In this notation, our goal is to show that

$$U_i(1) \le 1 - c_f(i).$$
 (C.12)

Now, if $U_i(1) \leq c_f(1)$, then since $c_f(1) \leq 1 - c_f(i) - \sum_{j < i} c_f(j) \leq 1 - c_f(i)$ by (4.1) of Definition 5, this immediately implies (C.12). Thus, for the remainder of the proof, we assume that $U_i(1) > c_f(1)$. In this case, it will be convenient to take $u_0 \in (0, 1)$ such that

$$U_i(1) = c_f(1)(u_0 - \log(u_0)).$$
(C.13)

First observe that since (4.7) holds for all j < i, we can apply Lemma 4.2 to get $\mathbb{E}[\mathcal{T}_{\mathsf{f}}(i) \mid \Lambda = \mathsf{f}] = \sum_{j < i} c_{\mathsf{f}}(j)\mu_j$. On the other hand, by writing $\mathbb{E}[\mathcal{T}_{\mathsf{f}}(i) \mid \Lambda = \mathsf{f}]$ as a Riemann–Stieltjes integral, and applying integration by parts, we get that

$$\sum_{j < i} c_{\mathsf{f}}(j)\mu_j = \mathbb{E}[\mathcal{T}_{\mathsf{f}}(i) \mid \Lambda = \mathsf{f}] = \Pr[0 \le \mathcal{T}_{\mathsf{f}}(i) \le 1 \mid \Lambda = \mathsf{f}] - \int_0^1 \Pr[0 \le \mathcal{T}_{\mathsf{f}}(i) \le \tau \mid \Lambda = \mathsf{f}]d\tau$$
$$= U_i(1) - \int_0^1 U_i(\tau)d\tau.$$
(C.14)

Our goal is to upper bound the integral in (C.14). In order to do so, observe that by Lemma C.2, (4.6) holds for *i*. As a result, after rewriting this, we know that for all $0 < \tau \le 1/2$,

$$U_i(1-\tau) \le U_i(\tau) - c_{\mathsf{f}}(1) \log\left(\frac{U_i(\tau)}{c_{\mathsf{f}}(1)}\right) \tag{C.15}$$

Now, define

$$\tau_0 = \begin{cases} \min\{\tau \in (0, 1/2] : U_i(\tau) \ge c_{\mathsf{f}}(1) \cdot u_0\} & \text{if } U_i(1/2) \ge c_{\mathsf{f}}(1) \cdot u_0\\ \frac{1}{2} & \text{if } U_i(1/2) < c_{\mathsf{f}}(1) \cdot u_0. \end{cases}$$
(C.16)

We assume that $\tau_0 < 1/2$, as $\tau_0 = 1/2$ is an edge case that is handled easily. By applying a change of variables, followed by (C.15),

$$\int_{\tau_0}^{1/2} U_i(\tau) d\tau + \int_{1/2}^{1-\tau_0} U_i(\tau) d\tau = \int_{\tau_0}^{1/2} U_i(\tau) d\tau + \int_{\tau_0}^{1/2} U_i(1-\tau) d\tau$$
$$\leq \int_{\tau_0}^{1/2} \left(2U_i(\tau) - c_{\mathsf{f}}(1) \log\left(\frac{U_i(\tau)}{c_{\mathsf{f}}(1)}\right) \right) d\tau \qquad (C.17)$$

More, by using the definition of τ_0 , together with (C.15) at $\tau = 1/2$,

$$U_i(\tau) \le c_{\mathsf{f}}(1)u_0 \text{ if } \tau \in [0, \tau_0)$$
 (C.18)

$$c_{\mathsf{f}}(1) \cdot u_0 \le U_i(\tau) \le U_i(1/2) \le c_{\mathsf{f}}(1) \text{ if } \tau \in [\tau_0, 1/2]$$
 (C.19)

$$U_i(\tau) \le U_i(1) \text{ if } \tau \in [1 - \tau_0, 1]$$
 (C.20)

Thus, using (C.18) and (C.20), followed by (C.13) and (C.17),

$$\int_0^1 U_i(\tau) d\tau \le \tau_0 c_{\mathsf{f}}(1) u_0 + \int_{\tau_0}^{1/2} U_i(\tau) d\tau + \int_{1/2}^{1-\tau_0} U_i(\tau) d\tau + U_i(1) \tau_0$$

$$\leq \tau_0(2c_{\mathsf{f}}(1) - c_{\mathsf{f}}(1)\log(u_0)) + \int_{\tau_0}^{1/2} \left(2U_i(\tau) - c_{\mathsf{f}}(1)\log\left(\frac{U_i(\tau)}{c_{\mathsf{f}}(1)}\right)\right) d\tau.$$

$$\leq \tau_0(2c_{\mathsf{f}}(1) - c_{\mathsf{f}}(1)\log(u_0)) + (1/2 - \tau_0)\max\{2c_{\mathsf{f}}(1) - c_{\mathsf{f}}(1)\log(u_0), 2c_{\mathsf{f}}(1)\},$$

where the final line uses (C.19) combined with the convexity of $z \rightarrow 2z - c_{\rm f}(1) \log(z/c_{\rm f}(1))$ on (0, 1).

We verify that (C.12) holds by handling both cases of the maximum. If $2c_{f}(1) - c_{f}(1) \log(u_{0}) \leq 2c_{f}(1)$, then $\int_{0}^{1} U_{i}(\tau) d\tau \leq c_{f}(i)$, and so $U_{i}(1) \leq c_{f}(1) + \sum_{j < i} c_{f}(j) \mu_{j} \leq 1 - c_{f}(i)$, where the last line applies (4.1) of Definition 5.

On the other hand, if $2c_f(1) < 2c_f(1) - c_f(1)\log(u_0)$, then $\int_0^1 U_i(\tau)d\tau \le c_f(1)u_0 - \frac{c_f(1)}{2}\log(u_0)$, and so

$$U_i(1) = c_{\mathsf{f}}(1)u_0 - c_{\mathsf{f}}(1)\log(u_0) \le \sum_{j \le i} c_{\mathsf{f}}(j)\mu_j + c_{\mathsf{f}}(1)u_0 - \frac{c_{\mathsf{f}}(1)}{2}\log(u_0),$$

which implies $u_0 \ge \exp\left(-\frac{2}{c_{\mathsf{f}}(1)}\sum_{j<i}c_{\mathsf{f}}(j)\mu_j\right)$. Thus, since $z \to z - \log(z)$ is non-increasing on (0,1), we get that $U_i(1) \le 2\sum_{j<i}c_{\mathsf{f}}(j)\mu_j + c_{\mathsf{f}}(1)\exp\left(-\frac{2}{c_{\mathsf{f}}(1)}\sum_{j<i}c_{\mathsf{f}}(j)\mu_j\right) \le 1 - c_{\mathsf{f}}(i)$, where the final inequality applies (4.2) of Definition 5.

C.2 Proof of Proposition 4.3

Recall that $\phi(z) := \frac{4}{9} - \frac{2z}{9}$ for each $z \in [0, 1]$. Clearly, ϕ is decreasing and continuous, and $(\phi(z) + \phi(1-z))/2 = 1/3$ for each $z \in [0, 1]$. We verify the remaining properties of Proposition 4.3 in order. Note that for each $z \in [0, 1]$,

$$\phi(z) + \phi(0) + \int_0^z \phi(\tau) d\tau = \left(\frac{4}{9} - \frac{2z}{9}\right) + \frac{4}{9} + \left(\frac{4z}{9} - \frac{z^2}{9}\right) = \frac{8}{9} + \frac{2z}{9} - \frac{z^2}{9} \le 1$$

where the inequality holds since the function of z is maximized at z = 1. Thus, (4.16) holds. Similarly,

$$\left(\phi(z) + 2\int_0^z \phi(\tau)d\tau\right) + \phi(0)\exp\left(-\frac{2\int_0^z \phi(\tau)d\tau}{\phi(0)}\right) = \left(\frac{22}{9} + \frac{2z}{9} - \frac{z^2}{9}\right) + \frac{4}{9}\exp\left(-\frac{9}{2}\left(\frac{4z}{9} - \frac{z^2}{9}\right)\right) \le 1$$

where the inequality holds since the function of z is maximized at z = 1. Thus, (4.17) holds, and so the proof is complete.

C.3 Proof of Lemma 4.4

Recall that we have assumed $\sum_{i=1}^{n} \mu_i = 1$, and $\mu_i > 0$ for each $i \in [n]$. We first argue that $(c_{\mathsf{f}}(i), c_{\mathsf{b}}(i))_{i=1}^n$ satisfy Definition 5. Now, $(c_{\mathsf{f}}(i))_{i=1}^n$ (respectively, $(c_{\mathsf{b}}(i))_{i=1}^n$) is non-increasing (respectively, non-decreasing), due to the fact that ϕ is decreasing, as claimed in Proposition 4.3. We next verify (4.1) and (4.2) of Definition 5 hold, beginning with $\sigma = \mathsf{f}$. Observe that

$$\sum_{j < i} c_{\mathbf{f}}(j) \mu_j = \sum_{j < i} \int_{\mu_{\mathbf{f}}(j)}^{\mu_{\mathbf{f}}(j) + \mu_j} \phi(\tau) d\tau = \int_0^{\mu_{\mathbf{f}}(i)} \phi(\tau) d\tau.$$
(C.21)

On the other hand, since ϕ is a decreasing function,

$$c_{\mathsf{f}}(i) = \int_{\mu_{\mathsf{f}}(i)}^{\mu_{\mathsf{f}}(i)+\mu_{i}} \frac{\phi(\tau)d\tau}{\mu_{i}} \le \phi(\mu_{\mathsf{f}}(i)), \text{ and } c_{\mathsf{f}}(1) = \int_{0}^{\mu_{1}} \frac{\phi(\tau)d\tau}{\mu_{1}} \le \phi(0).$$
(C.22)

Finally, using (C.21) and (C.22), together with the fact that $z \to z \exp\left(-\frac{2\int_0^{\mu_f(i)}\phi(\tau)d\tau}{z}\right)$ is increasing on $z \in (0,1)$,

$$c_{\mathsf{f}}(1) \exp\left(-\frac{2\sum_{j < i} c_{\mathsf{f}}(j)}{c_{\mathsf{f}}(1)}\right) = c_{\mathsf{f}}(1) \exp\left(-\frac{2\int_{0}^{\mu_{\mathsf{f}}(i)} \phi(\tau) d\tau}{c_{\mathsf{f}}(1)}\right) \le \phi(0) \exp\left(-\frac{2\int_{0}^{\mu_{\mathsf{f}}(i)} \phi(\tau) d\tau}{\phi(0)}\right) \tag{C.23}$$

By combining (C.21), (C.22), and (C.23), we get that

$$\begin{aligned} c_{\mathsf{f}}(i) + 2\sum_{j < i} c_{\mathsf{f}}(j)\mu_{j} + c_{\mathsf{f}}(1) \exp\left(-\frac{2\sum_{j < i} c_{\mathsf{f}}(j)}{c_{\mathsf{f}}(1)}\right) \\ & \leq \phi(\mu_{\mathsf{f}}(i)) + 2\int_{0}^{\mu_{\mathsf{f}}(i)} \phi(\tau)d\tau + \phi(0) \exp\left(-\frac{2\int_{0}^{\mu_{\mathsf{f}}(i)} \phi(\tau)d\tau}{\phi(0)}\right) \leq 1, \end{aligned}$$

where the last inequality follows from (4.17) of Proposition 4.3. Thus, (4.2) of Definition 5 holds for $\sigma = f$. By using (C.22) and (C.22), we can derive (4.1) of Definition 5 by using (4.16) of Proposition 4.3. Similar arguments also apply to $\sigma = b$. Thus, $(c_f(i), c_b(i))_{i=1}^n$ satisfy Definition 5.

Observe now that for any $i \in [n]$, we have that $\mu_{\mathsf{b}}(i) + \mu_i = 1 - \mu_{\mathsf{f}}(i)$. Thus,

$$\frac{c_{f}(i) + c_{b}(i)}{2} = \int_{\mu_{f}(i)}^{\mu_{f}(i) + \mu_{i}} \frac{\phi(\tau)}{2\mu_{i}} d\tau + \int_{\mu_{b}(i)}^{\mu_{b}(i) + \mu_{i}} \frac{\phi(\tau)}{2\mu_{i}} d\tau \\
= \int_{\mu_{f}(i)}^{\mu_{f}(i) + \mu_{i}} \frac{\phi(\tau)}{2\mu_{i}} d\tau + \int_{1 - \mu_{f}(i) - \mu_{i}}^{1 - \mu_{f}(i)} \frac{\phi(\tau)}{2\mu_{i}} d\tau \\
= \frac{1}{\mu_{i}} \left(\int_{\mu_{f}(i)}^{\mu_{f}(i) + \mu_{i}} \frac{\phi(\tau) + \phi(1 - \tau)}{2} d\tau \right) = \frac{1}{3},$$

where the third equality applies a change of variables, and the last equality applies Proposition 4.3. Thus, $\min_{i \in [n]} \frac{c_{f}(i) + c_{b}(i)}{2} = \frac{1}{3}$, and so the proof is complete.